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Improved lumped models for transient heat conduction in a slab with temperature-dependent thermal conductivity

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Abstract

This work reports improved lumped-parameter models for transient heat conduction in a slab with temperature-dependent thermal conductivity. The improved lumped models are obtained through two point Hermite approximations for integrals. For linearly temperature-dependent thermal conductivity, it is shown by comparison with numerical solution of the original distributed parameter model that the higher order lumped model ($H_{1,1}/H_{0,0}$ approximation) yields significant improvement of average temperature prediction over the classical lumped model. A unified Biot number limit depending on a single dimensionless parameter β is given both for cooling and heating processes.

Keywords: Transient heat conduction; Lumped models; Temperature-dependent thermal conductivity; Nonlinear heat conduction

1. Introduction

Accurate and comprehensive computational techniques such as finite difference, finite volume, and finite element methods can be applied to solve partial differential equations that model transport phenomena by distributed parameter formulations. However, in the analysis of complex industrial processes or systems, engineers sometimes rather prefer to predict and control the system behaviour using a simpler or simplified model that approximates accurately the original distributed parameter formulation but involves fewer state variables and consequently less equations to be solved. Model reduction techniques have received increasing attention in recent years, both in the applied mathematics community and in various application areas such as thermal systems, chemical engineering, electronic systems, and building simulation [1–9].

In the analysis of complex thermal systems, the lumped parameter formulation is a powerful engineering tool when a simplified model of the transient heat conduction is sought. As a rule of thumb, the classical lumped parameter approach, where uniform temperature is assumed within the region, is in general

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Nomenclature

b	temperature-dependent coefficient of thermal conductivity
Bi	Biot number ($= hL/k$)
c_p	specific heat
h	convective heat transfer coefficient
k	thermal conductivity
L	thickness of the slab
T	temperature
T_{i}	initial temperature
T_∞	environmental fluid temperature
T_0	reference temperature
t	time
Greek Letters	
α_0	reference thermal diffusivity ($= k_0/\rho c_p$)
β	dimensionless temperature-dependent coefficient of thermal conductivity
λ	dimensionless thermal conductivity
η	dimensionless spatial coordinate
ρ	density
τ	dimensionless time ($= \alpha_0 t/L^2$)
θ	dimensionless temperature ($= (T - T_\infty)/(T_{\mathrm{i}} - T_\infty)$)
Subscripts	
0	reference
av	average

restricted to problems with Biot number less than 0.1. In most engineering applications, however, the Biot number is much higher [10]. In other words, the moderate to low temperature gradient assumption is not reasonable in such applications, thus more accurate approach should be adopted. Improved lumped models have been developed by different approaches [11–19]. Nonlinear boundary conditions have been also investigated [20,21,16]. Cotta and Mikhailov [13] presented a systematic formalism to provide improved lumped parameter formulations for steady and transient heat conduction problems based on Hermite approximations for integrals that define averaged temperatures and heat fluxes. This approach has been shown to be efficient in a variety of practical applications [11,12,14,22–24]. Recently, Alhama and Zueco [25] applied for the first time a lumped model to nonlinear heat conduction problem of a slab with linearly temperature-dependent thermal conductivity. They studied both the cooling and heating processes and presented solutions as a function of the Biot number and a dimensionless parameter representing the heating or cooling process.

In this work, we present improved lumped models for transient heat conduction in a slab with cubically temperature-dependent thermal conductivity and subject to convective cooling or heating. The situation of simultaneous specific heat variation with temperature is not addressed in this work as a different technique is required for the development of improved lumped formulations [26]. The proposed lumped models are obtained through two point Hermite approximations for integrals [27,13]. The boundary temperature is related to the average temperature through fourth-order linear equations which are solved readily to close the ordinary differential equations for the average temperature. For linearly temperature-dependent thermal conductivity, the influence of the Biot number and the linearly temperature-dependent coefficient thermal conductivity on the accuracy of the classical and improved lumped models is investigated by comparison with finite difference solution of the original distributed parameter formulation. A unified Biot number limit is obtained as a function of the linear dependence coefficient β.

2. The mathematical formulation

Let us consider the transient heat conduction in a slab with temperature-dependent thermal conductivity and subject to convective cooling or heating. The wall is of thickness L and initially at a uniform temperature T_i. At $t = 0$, the wall is exposed to an environment of a constant fluid temperature T_∞ with a constant convective heat transfer coefficient h. The opposite side of the slab is adiabatic. The mathematical formulation of the problem is given by

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right), \quad \text{in } 0 < x < L \quad \text{for } t > 0, \tag{1}$$

with initial and boundary conditions taken as

$$T(x,0) = T_i, \quad \text{in } 0 < x < L, \quad \text{at } t = 0, \tag{2}$$

$$\frac{\partial T}{\partial x} = 0, \quad \text{at } x = 0 \quad \text{for } t > 0, \tag{3}$$

$$-k(T)\frac{\partial T}{\partial x} = h(T - T_\infty), \quad \text{at } x = L \quad \text{for } t > 0. \tag{4}$$

The thermal conductivity is given by a third-order polynomial

$$k(T) = k_0\left[1 + b_1(T - T_0) + b_2(T - T_0)^2 + b_3(T - T_0)^3\right], \tag{5}$$

where k_0 is the thermal conductivity at a reference temperature T_0, which is taken as the same as T_∞ for convenience.

The mathematical formulation (1)–(4) can now be rewritten in dimensionless form as

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \eta}\left(\lambda(\theta)\frac{\partial \theta}{\partial \eta}\right), \quad \text{in } 0 < \eta < 1 \quad \text{for } \tau > 0, \tag{6}$$

$$\theta(\eta,0) = 1, \quad \text{in } 0 < \eta < 1, \quad \text{at } \tau = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \eta} = 0, \quad \text{at } \eta = 0 \quad \text{for } \tau > 0, \tag{8}$$

$$-\lambda(\theta)\frac{\partial \theta}{\partial \eta} = Bi\theta, \quad \text{at } \eta = 1 \quad \text{for } \tau > 0, \tag{9}$$

The dimensionless thermal conductivity is given by

$$\lambda(\theta) = 1 + \beta_1\theta + \beta_2\theta^2 + \beta_3\theta^3. \tag{10}$$

It can be seen that the problem is governed by the Biot number, Bi, and the coefficients of the dimensionless thermal conductivity, β_1, β_2, and β_3.

3. Lumped models

We first introduce the spatially averaged dimensionless temperature as follows:

$$\theta_{av}(\tau) = \int_0^1 \theta(\eta,\tau)\mathrm{d}\eta. \tag{11}$$

We operate Eq. (6) by $\int_0^1 \mathrm{d}\eta$, using the definition of average temperature, Eq. (11), we get

$$\frac{\mathrm{d}\theta_{av}(\tau)}{\mathrm{d}\tau} = \frac{\lambda(\theta)\partial\theta}{\partial\eta}\bigg|_{\eta=1} - \lambda(\theta)\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0}. \tag{12}$$

Now, the boundary conditions Eqs. (8) and (9) are used, we have

$$\frac{\mathrm{d}\theta_{\mathrm{av}}(\tau)}{\mathrm{d}\tau} = -Bi\theta(1,\tau). \tag{13}$$

Eq. (13) is an equivalent integro-differential formulation of the mathematical model, Eqs. (6)–(9), with no approximation involved.

Supposing that the temperature gradient is sufficiently uniform over the whole spatial solution domain, the classical lumped system analysis (CLSA) is based on the assumption that the boundary temperatures can be reasonably well approximated by the average temperature, as

$$\theta(0,\tau) \cong \theta(1,\tau) \cong \theta_{\mathrm{av}}(\tau),$$

which leads to the classical lumped model

$$\frac{\mathrm{d}\theta_{\mathrm{av}}(\tau)}{\mathrm{d}\tau} = -Bi\theta_{\mathrm{av}}(\tau) \tag{14}$$

and to be solved with the initial condition for the average temperature

$$\theta_{\mathrm{av}}(0) = 1. \tag{15}$$

It can be seen that the classical model shows no influence of the temperature-dependent thermal conductivity.

In an attempt to enhance the approximation approach of the classical lumped model, we develop improved lumped models by providing better relations between the boundary temperature and the average temperature, based on Hermite-type approximations for integrals that define the average temperature and the heat flux. The general Hermite approximation for an integral, based on the values of the integrand and its derivatives at the integration limits, is written in the following form [27]:

$$\int_a^b y(x)\mathrm{d}x = \sum_{v=0}^{\alpha} C_v y^{(v)}(a) + \sum_{v=0}^{\beta} D_v y^{(v)}(b),$$

where $y(x)$ and its derivatives $y^{(v)}(x)$ are defined for all $x \in (a,b)$. It is assumed that the numerical values of $y^{(v)}(a)$ for $v = 0,1,\ldots,\alpha$, and $y^{(v)}(b)$ for $v = 0,1,\ldots,\beta$ are available. The general expression for the $H_{\alpha,\beta}$ approximation is given by

$$\int_a^b y(x)\mathrm{d}x = \sum_{nu=0}^{\alpha} C_v(\alpha,\beta)h^{\alpha+1}y^{(v)}(a) + \sum_{nu=0}^{\beta} C_v(\beta,\alpha)h^{\alpha+1}y^{(v)}(b) + \mathrm{O}(h^{\alpha+\beta+3}),$$

where $h = b - a$, and

$$C_v(\alpha,\beta) = \frac{(\alpha+1)!(\alpha+\beta+1-v)!}{(v+1)!(\alpha-v)!(\alpha+\beta+2)!}.$$

We first employ the plain trapezoidal rule in the integrals for both average temperature and average heat flux ($H_{0,0}/H_{0,0}$ approximation), in the form

$$\theta_{\mathrm{av}}(\tau) \cong \frac{1}{2}[\theta(0,\tau) + \theta(1,\tau)], \tag{16}$$

$$\int_0^1 \frac{\partial\theta(\eta,\tau)}{\partial\eta}\mathrm{d}\eta = \theta(1,\tau) - \theta(0,\tau) \cong \frac{1}{2}\left[\frac{\partial\theta}{\partial\eta}\Big|_{\eta=0} + \frac{\partial\theta}{\partial\eta}\Big|_{\eta=1}\right]. \tag{17}$$

The boundary conditions (8) and (9) are substituted into Eq. (17) to yield

$$\theta(1,\tau) - \theta(0,\tau) = -\frac{Bi}{2}\theta(1,\tau). \tag{18}$$

The boundary temperature $\theta(0,\tau)$ is solved from Eq. (16) and substituted into Eq. (18) and we obtain an equation that relates $\theta(1,\tau)$ to $\theta_{\mathrm{av}}(\tau)$

$$4\beta_3\theta(1,\tau)^4+4(\beta_2-\beta_3\theta_{av}(\tau))\theta(1,\tau)^3+4(\beta_1-\beta_2\theta_{av}(\tau))(\theta(1,\tau)^2)+(4+Bi-4\beta_1\theta_{av}(\tau))\theta(1,\tau)-4\theta_{av}(\tau)=0. \tag{19}$$

Then we further improve the lumped model by employing two-side corrected trapezoidal rule in the integral for average temperature, in the form

$$\theta_{av}(\tau) \cong \frac{1}{2}[\theta(0,\tau)+\theta(1,\tau)] + \frac{1}{12}\left[\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0} - \frac{\partial\theta}{\partial\eta}\bigg|_{\eta=1}\right]. \tag{20}$$

The boundary conditions (8) and (9) are substituted into Eq. (20) to yield

$$\theta_{av}(\tau) \cong \frac{1}{2}[\theta(0,\tau)+\theta(1,\tau)] + \frac{Bi}{12}\theta(1,\tau), \tag{21}$$

while keeping the plain trapezoidal rule in integral for heat flux ($H_{1,1}/H_{0,0}$ approximation).

The boundary temperature $\theta(0,\tau)$ is solved from Eq. (21) and substituted into Eq. (18), we obtain an equation that relates $\theta(1,\tau)$ to $\theta_{av}(\tau)$

$$3\beta_3\theta(1,\tau)^4+3(\beta_2-\beta_3\theta_{av}(\tau))\theta(1,\tau)^3+3(\beta_1-\beta_2\theta_{av}(\tau))(\theta(1,\tau)^2)+(3+Bi-3\beta_1\theta_{av}(\tau))\theta(1,\tau)-3\theta_{av}(\tau)=0. \tag{22}$$

Although explicit analytical solutions of the fourth-order polynomial equations (19) and (22) can be found readily by using a symbolic-numerical software such as *Mathematica*, we will limit the present study to the particular case of linear dependence of thermal conductivity with temperature, that is, $\beta_2=\beta_3=0$. In this case, we replace β_1 with β for convenience. Eq. (19) reduces to

$$4\beta\theta(1,\tau)^2 + (4+Bi-4\beta\theta_{av}(\tau))\theta(1,\tau) - 4\theta_{av}(\tau) = 0. \tag{23}$$

Analytical solution of Eq. (23) is readily obtained and then used to close the ordinary differential Eq. (13) for the average temperature, to be solved with the initial condition Eq. (15), providing the $H_{0,0}/H_{0,0}$ model

$$\frac{d\theta_{av}(\tau)}{d\tau} = -\frac{Bi}{8\beta}\left(-4-Bi+4\beta\theta_{av}(\tau)+\sqrt{64\beta\theta_{av}(\tau)+(4+Bi-4\beta\theta_{av}(\tau))^2}\right). \tag{24}$$

Similarly, when $\beta_2=\beta_3=0$, Eq. (22) reduces to

$$3\beta\theta(1,\tau)^2 + (3+Bi-3\beta\theta_{av}(\tau))\theta(1,\tau) - 3\theta_{av}(\tau) = 0. \tag{25}$$

Similarly, the analytical solution of Eq. (25) is obtained and used to close the ordinary differential Eq. (13) for the average temperature, to be solved with the initial condition Eq. (15), providing the $H_{1,1}/H_{0,0}$ model

$$\frac{d\theta_{av}(\tau)}{d\tau} = -\frac{Bi}{6\beta}\left(-3-Bi+3\beta\theta_{av}(\tau)+\sqrt{36\beta\theta_{av}(\tau)+(3+Bi-3\beta\theta_{av}(\tau))^2}\right). \tag{26}$$

4. Discussions on cooling and heating processes

Writing $k(T)=k_1+k_2(T)$, Alhama and Zueco [25] identified four different kinds of problem that may occur: (i) a heating process with a positive temperature-dependent coefficient, $k_2>0$; (ii) a heating process with $k_2<0$; (iii) a cooling process with $k_2>0$ and (iv) a cooling process with $k_2<0$. They established that the universal mean Biot number limit for applying the lumped model can be expressed as a function of the dimensionless number $\kappa=(k_{max}-k_{min})/k_m$, and the kind of process (cooling or heating), with $k_m=(k_{min}+k_{max})/2$.

In what follows, we are going to show that under proper choice of dimensionless parameters, the four kinds of problem can be reduced to two kinds of problem: (i) $\beta>0$, representing cooling with a positive temperature-dependent coefficient $b>0$ or heating with $b<0$ and (ii) $\beta<0$, representing cooling with $b<0$ or heating with $b>0$. The main difference between Alhama–Zueco's analysis and ours lies in the choice of the reference temperature. While Alhama and Zueco [25] always use the minimum temperature T_{min} as the reference temperature, we always use the surrounding fluid temperature T_∞ as the reference temperature whether cooling or heating. For a linearly temperature-dependent thermal conductivity,

$$k(T) = k_\infty(1 + b(T - T_\infty)),$$

we have for a cooling process ($T_i > T_\infty$) with a positive temperature-dependent coefficient ($b > 0$)

$$\lambda(\theta) = \frac{k(T)}{k_\infty} = 1 + b(T_i - T_\infty)\theta = 1 + \beta\theta$$

thus $\beta = b(T_i - T_\infty) > 0$. For a cooling process with $b < 0$, we have

$$\lambda(\theta) = 1 + b(T_i - T_\infty)\theta = 1 + \beta\theta,$$

with $\beta = b(T_i - T_\infty) < 0$. For a heating process ($T_i < T_\infty$) with $b > 0$, we have

$$\lambda(\theta) = 1 + b(T_i - T_\infty)\theta = 1 + \beta\theta$$

with $\beta = b(T_i - T_\infty) < 0$. For a heating process with $b < 0$, we have

$$\lambda(\theta) = 1 + b(T_i - T_\infty)\theta = 1 + \beta\theta$$

with $\beta = b(T_i - T_\infty) > 0$.

It can be seen that the four kinds of problem identified by Alhama and Zueco [25] can be represented conveniently by only one dimensionless parameter β, with $\beta > 0$ representing cooling with $b > 0$ or heating with $b < 0$, and $\beta < 0$ representing cooling with $b < 0$ or heating with $b > 0$.

We proceed to examine the example problems given by Alhama and Zueco [25].

Problem 1. $T_i = 1$, $T_\infty = 0$, $k(T) = 0.9 + 0.2T$, $k_{\min} = 0.9$, $k_{\max} = 1.1$, $k_m = 1$, $\kappa = 0.2$.

By our analysis, $\theta = T$, $\theta_i = 1$, $\theta_\infty = 0$, $\lambda(\theta) = k(T)/k_\infty = 1 + (2/9)\theta$, $\beta = 2/9$.

Problem 2. $T_i = 1$, $T_\infty = 0$, $k(T) = 1.8 + 0.4T$, $k_{\min} = 1.8$, $k_{\max} = 2.2$, $k_m = 2$, $\kappa = 0.2$.

By our analysis, $\theta = T$, $\theta_i = 1$, $\theta_\infty = 0$, $\lambda(\theta) = k(T)/k_\infty = 1 + (2/9)\theta$, $\beta = 2/9$.

Problem 3. $T_i = 10$, $T_\infty = 0$, $k(T) = 0.9 + 0.02T$, $k_{\min} = 0.9$, $k_{\max} = 1.1$, $k_m = 1$, $\kappa = 0.2$.

By our analysis, $\theta = T/10$, $\theta_i = 1$, $\theta_\infty = 0$, $\lambda(\theta) = k(T)/k_\infty = 1 + (2/9)\theta$, $\beta = 2/9$.

The difference between Alhama–Zueco's and our analyses is shown when examining the heating processes with a positive temperature-dependent coefficient ($k_2 > 0$ or $b > 0$).

Problem 4. $T_i = 0$, $T_\infty = 1$, $k(T) = 0.9 + 0.2T$, $k_{\min} = 0.9$, $k_{\max} = 1.1$, $k_m = 1$, $\kappa = 0.2$.

By our analysis, $\theta = (T - 1)/(-1)$, $\theta_i = 1$, $\theta_\infty = 0$,

$$\lambda(\theta) = \frac{k(T)}{k_\infty} = \frac{0.9 + 0.2(-\theta + 1)}{1.1} = 1 - \frac{2}{11}\theta,$$

thus $\beta = 2/11$.

Problem 5. $T_i = 0$, $T_\infty = 1$, $k(T) = 1.8 + 0.4T$, $k_{\min} = 1.8$, $k_{\max} = 2.2$, $k_m = 2$, $\kappa = 0.2$.

By our analysis, $\theta = (T - 1)/(-1)$, $\theta_i = 1$, $\theta_\infty = 0$,

$$\lambda(\theta) = \frac{k(T)}{k_\infty} = \frac{1.8 + 0.4(-\theta + 1)}{2.2} = 1 - \frac{2}{11}\theta,$$

thus $\beta = 2/11$.

Problem 6. $T_i = 0$, $T_\infty = 10$, $k(T) = 0.9 + 0.02T$, $k_{\min} = 0.9$, $k_{\max} = 1.1$, $k_m = 1$, $\kappa = 0.2$.

By our analysis, $\theta = (T - 10)/(-10)$, $\theta_i = 1$, $\theta_\infty = 0$,

$$\lambda(\theta) = \frac{k(T)}{k_\infty} = \frac{0.9 + 0.02 \times 10(-\theta + 10)}{2.2} = 1 - \frac{2}{11}\theta,$$

thus $\beta = 2/11$.

It can be seen that Problems 1–3 reduce to a same dimensionless problem with $\beta = 2/9$ and Problems 4–6 reduce to another dimensionless problem with $\beta = -2/11$.

5. Numerical results and discussions

The solutions of classical and improved lumped models are shown in tabular and graphical forms in comparison with a reference finite difference solution of the original distributed model, Eqs. (6)–(9). The initial boundary value problem defined by Eqs. (6)–(9) is solved by using an implicit finite difference method, with a 201 nodes mesh in spatial discretization and a dimensionless time step of 0.00001 for all cases. Different values of the Biot number Bi and the parameter β are chosen so as to assess accuracy of the solutions given by lumped models.

In Table 1, it is presented a comparison of the dimensionless average temperatures obtained by lumped models and the reference finite difference solution of the original distributed parameter model at different values of time, for $Bi = 1.0$ and $\beta = 1.0$. As can be seen, the classical lumped model gives an error of 0.0681 at $\tau = 1.0$, while the $H_{0,0}/H_{0,0}$ model gives an error of 0.0137 at $\tau = 1.0$, and the $H_{1,1}/H_{0,0}$ model yields a maximum error less than 0.005 for all time values. Fig. 1 shows the comparison of the dimensionless average temperatures for $Bi = 2.5$ and $\beta = 0.5$. It can be seen that the solution give by the higher order improved lumped model ($H_{1,1}/H_{0,0}$) agrees quite well with the finite difference solution.

Table 1
Comparison of lumped models against finite difference solution for dimensionless average temperature $\theta_{av}(\tau)$ at different values of time

τ	FD Solution $Bi = 1.0$	CLSA $\beta = 1.0$	$H_{0,0}/H_{0,0}$	$H_{1,1}/H_{0,0}$
0.1	0.9150	0.9048	0.9157	0.9190
0.2	0.8406	0.8187	0.8389	0.8450
0.3	0.7730	0.7408	0.7689	0.7774
0.4	0.7113	0.6703	0.7050	0.7156
0.5	0.6548	0.6065	0.6466	0.6589
0.6	0.6031	0.5488	0.5934	0.6070
0.7	0.5557	0.4966	0.5447	0.5595
0.8	0.5123	0.4493	0.5002	0.5159
0.9	0.4725	0.4066	0.4595	0.4758
1.0	0.4359	0.3679	0.4222	0.4391
2.0	0.1985	0.1353	0.1838	0.1997
3.0	0.0925	0.0498	0.0813	0.0926
4.0	0.0436	0.0183	0.0363	0.0434
5.0	0.0207	0.0067	0.0163	0.0204

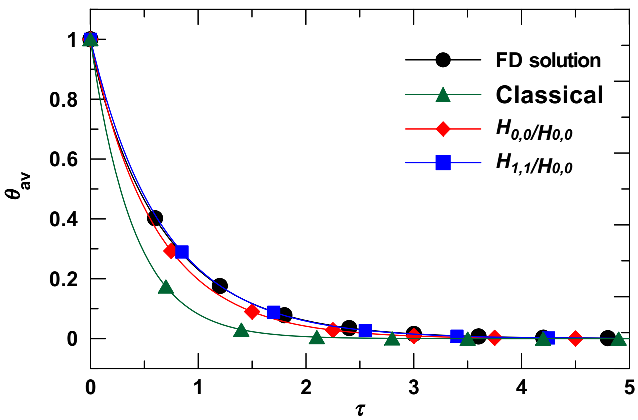

Fig. 1. Dimensionless temperature as a function of dimensionless time for $Bi = 2.5$ and $\beta = 0.5$.

The normalized error or dimensionless deviation of the average temperature is defined as follows:

$$e = \left|\frac{T_{\mathrm{av,lumped}}(t) - T_{\mathrm{av,FD}}(t)}{T_{\mathrm{i}} - T_{\infty}}\right| = \left|\theta_{\mathrm{av,lumped}}(\tau) - \theta_{\mathrm{av,FD}}(\tau)\right|.$$

The typical error distribution of the lumped model $H_{1,1}/H_{0,0}$ as a function of dimensionless time is shown in Fig. 2 for $\beta = 0.5$ and three Biot numbers, $Bi = 1.0, 1.5$, and 2.0. As can be seen, the error rises to maximum in early times, decays to zero in intermediate times and then rises again. The influence of the sign of β on the error distribution is shown in Fig. 3 for $Bi = 1.0$, $\beta = 0.2$ and -0.2. It can be seen that the pattern of the error distribution is similar for positive and negative β, while the maximum error for $\beta = -0.2$ is slightly higher than that for $\beta = 0.2$. The increase of the maximum error of the average temperature of the improved lumped model $H_{1,1}/H_{0,0}$ with increasing Biot number can be seen clearly in Fig. 4. In this work, the criterion for the application of the lumped model is defined as that the maximum normalized error is less than 0.01. The Biot number limit can be thus determined from Fig. 4, as shown by the broken line. The variation of the Biot number limit of the improved lumped model $H_{1,1}/H_{0,0}$ as a function of β is shown in Fig. 5. A linear regression of the data gives the straight line $Bi_{\mathrm{limit}} = 0.523\beta + 1.078$ for $-0.6 \leqslant \beta \leqslant 0.8$. The lumped model $H_{1,1}/H_{0,0}$ is expected to yield a maximum normalized error less 0.01 for $Bi < Bi_{\mathrm{limit}}(\beta)$, for a given β.

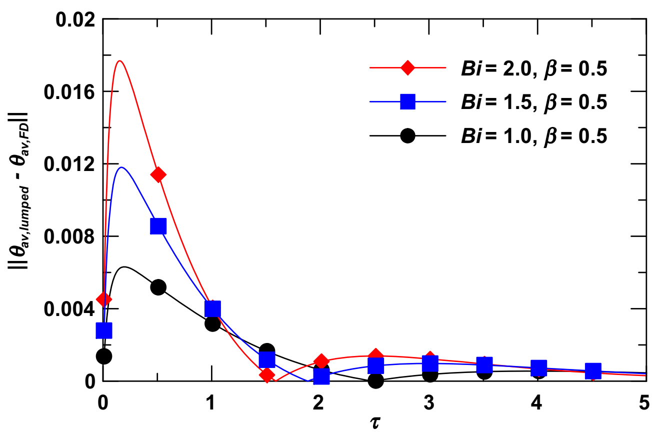

Fig. 2. The normalized error of the average dimensionless temperature by the $H_{1,1}/H_{0,0}$ model as a function of time for $\beta = 0.5$.

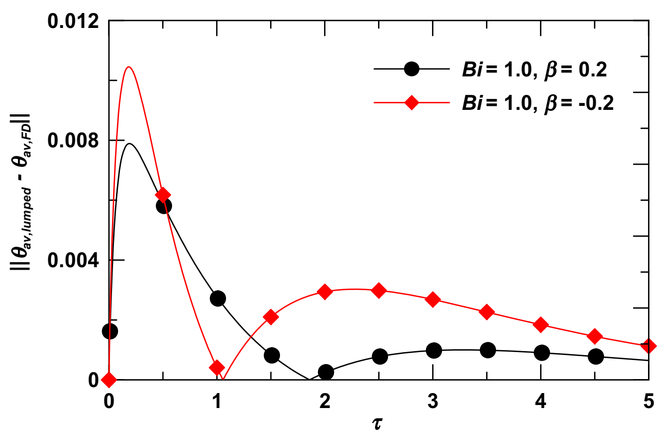

Fig. 3. The normalized error of the average dimensionless temperature by the $H_{1,1}/H_{0,0}$ model as a function of time for $Bi = 1.0$.

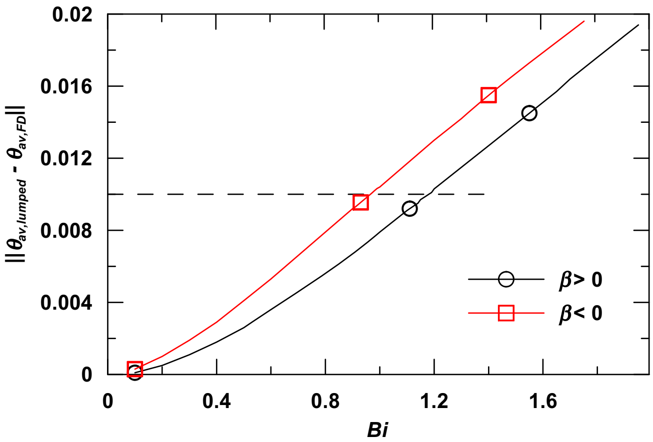

Fig. 4. The maximum error of the average dimensionless temperature by the $H_{1,1}/H_{0,0}$ model as a function of the Biot number for $\beta = 0.2$.

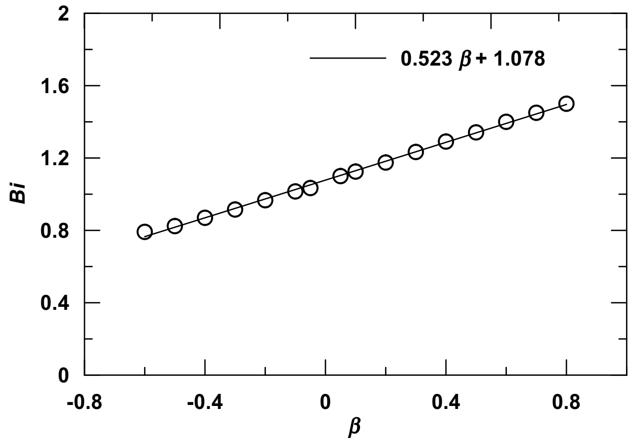

Fig. 5. The Biot number limit as a function of β.

6. Conclusions

Improved lumped parameter models are presented for transient heat conduction in a slab with cubicly temperature-dependent thermal conductivity and subject to convective cooling or heating. The improved lumped models are obtained through two point Hermite approximations for integrals. For linearly temperature-dependent thermal conductivity, it is shown by comparison with numerical solution of the original distributed parameter model that the higher order lumped model ($H_{1,1}/H_{0,0}$ approximation) yields significant improvement of average temperature prediction over the classical lumped model. It is shown that the maximum relative error of the dimensionless average temperature is influenced predominantly by the Biot number. A unified Biot number limit is obtained as a function of the linear dependence coefficient β, $Bi_{\text{limit}} = 0.523\beta + 1.078$ for $-0.6 \leqslant \beta \leqslant 0.8$. The lumped model $H_{1,1}/H_{0,0}$ is expected to yield maximum normalized error less 0.01 for $Bi < Bi_{\text{limit}}(\beta)$, for a given β.

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