Vorticity statistics and the time scales of turbulent strain

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Time scales of turbulent strain activity, denoted as the strain persistence times of first and second order, are obtained from time-dependent expectation values and correlation functions of Lagrangian rate-of-strain eigenvalues taken in particularly defined statistical ensembles. Taking into account direct numerical simulation data, our approach relies on heuristic closure hypotheses which allow us to establish a connection between the statistics of vorticity and strain. It turns out that softly divergent prefactors correct the usual “1/s” strain time-scale estimate of standard turbulence phenomenology, in a way which is consistent with the phenomenon of vorticity intermittency.

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I. INTRODUCTION

It is a point of reasonable consensus that further progress in the statistical theory of turbulence has been hampered in great part due to the fact that one of its phenomenological pillars, the Kolmogorov-Richardson cascade, is actually a longstanding open issue. The usual assumption of eddy stretching as the Kolmogorov-Richardson cascade, is actually a longstanding part due to the fact that one of its phenomenological pillars, the statistical theory of turbulence has been hampered in great variance with experimental and numerical observations [9]. Variance of strong vorticity fields and the phenomenon of turbulent intermittency.

Having in mind the above difficulties and relying more on computational data, our approach relies on heuristic closure hypotheses which allow us to establish a connection between the statistics of vorticity and strain. It turns out that softly divergent prefactors correct the usual “1/s” strain time-scale estimate of standard turbulence phenomenology, in a way which is consistent with the phenomenon of vorticity intermittency.

This paper is organized as follows. In the next section we address formal definitions of the strain persistence times and discuss, by means of a straightforward closure scheme motivated in great part by the analysis of direct numerical simulation (DNS) data, their relation to single-point vorticity statistics. In Sec. III, we verify, in the DNS context, that our analytical framework, devised to hold in principle in the small strain domain, incidentally holds for the whole range of strain strengths. In Sec. IV, we comment on our findings and point out directions of further research.

II. STRAIN PERSISTENCE TIMES

Let \( s_{ij} = (\partial_i v_j + \partial_j v_i) / 2 \) be the \((i,j)\) component of the Lagrangian rate-of-strain tensor. Recalling that \( s_{ij} \) is traceless due to incompressibility, call the only positive or the only negative eigenvalue of \( s_{ij} (t) \) by \( \bar{s}(t) \), a piecewise continuous function of time, as indicated in Fig. 1(a).

Independent turbulent flow realizations of \( \bar{s}(t) \) generated, for instance, from some set of random initial conditions at \( t \rightarrow -\infty \) constitute a large functional space \( S \). Take the ensemble \( \bar{\Lambda}_x \subset S \) of all the profiles \( \bar{s}(t) \), which have \( \bar{s}(0) = s \) for an arbitrarily prescribed eigenvalue \( s \). Alternatively, we define the related ensemble \( \Lambda_x \) of compactly supported functions \( s(t) \), which are identified to \( \bar{s}(t) \in \bar{\Lambda}_x \) in the largest neighborhood of \( t = 0 \) where \( \bar{s}(t) \) is continuous. The functions \( s(t) \) vanish out of these neighborhoods. See the sketches in Fig. 1(b).

In a more formal way \( \bar{\Lambda}_x \) is given as the ensemble of functions \( s(t) \) obtained from the one-to-one mapping

\[
\bar{\Lambda}_x \mapsto \Lambda_x, \quad \bar{s}(t) \mapsto s(t), \quad (2.1)
\]
given, for positive \( t \), by

\[
s(t) = \begin{cases} 
\bar{s}(t) & \text{if } \forall t' \in [0,t], \bar{s}(t')/s > 0 \\
0 & \text{if } \exists t' \in [0,t] \mid \bar{s}(t')/s < 0,
\end{cases} \quad (2.2)
\]

while for negative \( t \), the time interval \([0,t]\) is replaced, in (2.2), by \([t,0]\). Notwithstanding the mathematically rigorous language used in (2.1) and (2.2), it is important to keep in mind...
the essential heuristic-phenomenological flavor of the present work.

The rationale for the introduction of the ensemble $\Lambda_s$ is that its elements, i.e., the time-dependent strain eigenvalues $s(t)$, have all the same postulated “strain strength” $s \equiv s(0)$ and well-defined lifetimes, once they are compactly supported functions. Our task, therefore, is to investigate their characteristic time scales and to understand how they depend on $s$.

We emphasize that the just given definition of the ensemble $\Lambda_s$ is a strong idealization of dubious utility if the interest is to compute statistical averages of fluid dynamic observables out of experimental or numerical data. The essential difficulty here is that one should work, in principle, with a large functional space of turbulent flow realizations. However, by evoking ergodicity [10], a statistically equivalent ensemble $\Lambda_s$ can be introduced for practical purposes as it follows:

(i) Taking a single three-dimensional turbulent flow realization (e.g., the one obtained in a direct numerical simulation) recorded in a time interval of length $T$, pick up a number $N$ of (hopefully weakly correlated) Lagrangian trajectories;

(ii) An alternative set of time parametrizations of the Lagrangian trajectories is implemented by conventionally setting time $t = 0$ at $M$ equally spaced time instants along the dynamical evolution;

(iii) Define the ensemble $\bar{\Lambda}_s$ of $N \times M$ strain eigenvalue profiles $\bar{s}(t)$ obtained from the Lagrangian trajectories introduced in (i) and time-parametrized according to (ii);

(iv) Working with some assigned uncertainty $\delta s$ in the definition of $s$, introduce the ensemble $\Lambda_s$ of functions $s(t)$ derived from the profiles defined in (iii), as prescribed in (2.1) and (2.2).

Of particular importance in our considerations are the following time integrations over first- and second-order expectation values taken in $\Lambda_s$,

$$I_1(s) \equiv \int_{-\infty}^{0} dt \langle s(t) \rangle|_{\Lambda_s},$$

$$I_2(s) \equiv \int_{-\infty}^{0} dt \int_{-\infty}^{0} dt' \langle \delta s(t) \delta s(t') \rangle|_{\Lambda_s},$$

where

$$\delta s(t) \equiv s(t) - \langle s(t) \rangle|_{\Lambda_s}.$$  

Considering, for convenience, positive and negative $s$ as separate cases, strain persistence times of first and second order, $T^+_I(s)$ and $T^+_2(s)$, respectively, can be defined from (2.3) and (2.4), as

$$I_1(s) = \begin{cases} s T^+_I(s) & \text{for } s > 0 \\ s T^+_1(s) & \text{for } s < 0 \end{cases}$$

and

$$I_2(s) = \begin{cases} [s T^+_I(s)]^2 & \text{for } s > 0 \\ [s T^+_2(s)]^2 & \text{for } s < 0. \end{cases}$$

The central idea underlying our discussion is that analytical expressions for $T^+_I(s)$ and $T^+_2(s)$ can be derived from a statistical treatment of vorticity fluctuations, which are governed by the Lagrangian evolution equation [11],

$$\frac{d \omega_i}{dt} = s_{ij}(t) \omega_j + \epsilon_{ijk} \partial_j f_k + v \partial^2 \omega_i,$$

where $f_k$ and $v$ denote, respectively, the density of external force and the kinematic viscosity. Let $\check{n}(t)$ be the unit vector defined along the principal direction associated to the only positive or only negative eigenvalue $s(t)$ of the Lagrangian rate-of-strain tensor [the twofold orientation ambiguity of $\check{n}(t)$ is arbitrarily resolved]. One gets, after some simple algebra on Eq. (2.8),

$$\frac{d \omega^2}{dt} = 2s(t) \omega^2 + 2\omega \cdot \frac{d \check{n}}{dt} + \epsilon_{ijk} \partial_j f_k + v \partial^2 \omega_i,$$  

where $\omega(t) \equiv \omega(t) \cdot \check{n}(t)$. It is interesting to take the expectation value of Eq. (2.9) conditioned to a given time-dependent profile $s(t) \in \Lambda_s$. In other words, we change our focus to the alternative evolution equation

$$\frac{d \langle \omega^2 \rangle_{[s]}}{dt} = 2s(t) \langle \omega^2 \rangle_{[s]} + 2 \left[ \langle \omega \cdot \frac{d \check{n}}{dt} \rangle_{[s]} + \epsilon_{ijk} \langle \omega_i \partial_j f_k \rangle_{[s]} + v \langle \partial^2 \omega \rangle_{[s]} \right],$$

where $(\cdot \cdot \cdot)_{[s]}$ is the self-evident notation for the procedure of conditional averaging. It is clear that Eq. (2.10) is not closed. The second term on its right-hand-side, for instance, is related in an intricate way, through $d\check{n}/dt$, to the velocity.
fluctuations,
\[ \langle \omega^2(0) \rangle_{\Lambda_s} = \exp \left[ 2c + 2\alpha \int_{-\infty}^{0} dt \langle s(t) \rangle_{\Lambda_s} \right. \\
\left. + 2\beta \int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' \langle \delta s(t) \delta s(t') \rangle_{\Lambda_s} \right], \]  
(2.18)

Using Eqs. (2.3) to (2.7), the right-hand side of Eq. (2.18) can be rewritten as
\[ \langle \omega^2(0) \rangle_{\Lambda_s} = \begin{cases} 
\exp \left[ 2c + 2\alpha s T_1^+(s) + 2\beta [s T_1^+(s)]^2 \right] & \text{for } s > 0 \\
\exp \left[ 2c + 2\alpha s T_1^-(s) + 2\beta [s T_1^-(s)]^2 \right] & \text{for } s < 0.
\end{cases} \]  
(2.19)

It is also convenient to express, up to first order in a power series of \( s \), the standard deviation of \( \omega(0) \) in the ensemble \( \Lambda_s \) as
\[ \sigma_{\omega}(s) \equiv \sqrt{\langle \omega^2(0) \rangle_{\Lambda_s}} \]
(2.20)

where \( a \), \( b_+ \), and \( b_- \) are arbitrary coefficients, and we have used that \( \tilde{\omega} \) and \( \tilde{n} \) are independent random variables in \( \Lambda_s \), so that \( \langle \omega(0) \rangle_{\Lambda_s} = 0 \). As reported in the next section, the relevance of the above expansion is established in a purely empirical way (that is, from the analysis of DNS data) with the surprising result, still in need of theoretical understanding, that there are no higher order corrections to Eq. (2.20) even for reasonably large values of the strain eigenvalue \( s \).

As stated (in rephrased form) in the introductory section, we do not expect vanishing strain to have any effect on the statistics of vorticity. Requiring, therefore, that
\[ \lim_{s \to 0} [s T_{1+}^\pm(s)] = \lim_{s \to 0} [s T_{1-}^\pm(s)] = 0, \]  
(2.21)

it follows from Eqs. (2.19) and (2.20) that \( \exp(c) = a \).

Recalling, now, the time-reversal symmetry of the fluid dynamic equations in the absence of forcing and dissipation terms, a meaningful approximation for the description of inertial range processes, we assume that \( T_1^+(s) \propto T_1^-(s) \) and \( T_2^+(s) \propto T_2^-(s) \). We point out that this argument is not inconsistent at all with the dissipation anomaly postulated by the “zeroth law” of turbulencen, that is, that the fact that energy dissipation rate per unit volume is finite in the inviscid limit \( \nu \to 0 \) [11,14–17]. The situation here is analogous to the issue on the coexistence of the second law of thermodynamics with microscopic reversibility in the statistical mechanics context. Introducing a pair of even functions of \( s \), \( F_1(s) \) and \( F_2(s) \), and proportionality constants \( g \) and \( g' \), we may write, thus, without loss of further generality, that
\[ T_1^+(s) = \frac{1}{\alpha s} \ln F_1(s), \quad T_1^-(s) = -\frac{g}{\alpha s} \ln F_1(s), \]  
(2.22)

\[ T_2^+(s) = \frac{1}{\beta s^2} \ln F_2(s), \quad T_2^-(s) = \frac{g'}{\beta s^2} \ln F_2(s). \]  
(2.23)
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III. ANALYSIS OF DNS DATA

We have computed statistical averages of fluid dynamic observables with the help of the direct numerical simulation (DNS) database available from the turbulence research group at Johns Hopkins University [19]. An homogeneous and isotropic turbulent flow with Taylor-based Reynolds number $R_t \approx 433$ is simulated in a periodic cube of linear dimension $L = 2\pi$ modeled as a grid of $1024^3$ lattice points. Viscosity and time step parameters are, respectively, $\nu = 1.85 \times 10^{-4}$ and $\Delta t = 2 \times 10^{-4}$ (the complete simulation record corresponds to around one large eddy turnover time, $T_0 = 2^{10} \times 10 \times \Delta t$). Further simulation details can be found in Refs. [20,21].

Statistical samples were produced in two different ways, according to the particular expectation values we were interested to evaluate:

(i) In order to compute $\sigma_\omega(s) = \sqrt{\langle \omega^2(0) \rangle_L}$, vorticity vectors and rate-of-strain tensors were defined from the velocity gradients taken at grid points $2\pi(i, 8j, 8k)/1024$ where $0 \leq i < 1024$ and $0 \leq j, k < 128$ are integer numbers, for $10^3$ frames of equally time-spaced flow configurations. Bins of variable sizes were considered for the sets of positive and negative rate-of-strain eigenvalues $s$. While essentially conventional, our particular bin size choice, $\delta s = 0.5$, proved to yield robust results at economical computational costs.

(ii) $I_1(s)$ and $I_2(s)$, as given in Eqs. (2.3) and (2.4), were computed from $10^3$ Lagrangian trajectories, each one consisting of $2^{10}$ time steps (which are separated in time by $2^{10} \times 10^{-3}$ arbitrary time units). The initial points of the Lagrangian trajectories are given by $2\pi(i, j, k)/10$, with $0 \leq i, j, k \leq 10$. The ensemble $\Lambda_s$, with uncertainty $\delta s = 0.1$, was generated through the procedure previously discussed in Sec. II.

In both of the above cases (i) and (ii), the eigenvalues and principal directions of the rate-of-strain tensor were computed through an efficient hybrid algorithm which combines direct analytical evaluation and the so-called QL algorithm [22].

As is clear from Fig. 2, the conditional expectation value $\sigma_\omega(s)$ is precisely, and surprisingly well, described by Eq. (2.20), with no additional corrections. At $s = 0$ we have $a = \sigma_\omega(0) = 3.6$. The slope parameters $b_+$ and

\[ a = \sigma_\omega(0) = 3.6. \]

The slope parameters $b_+$ and $b_-$ are, therefore, likely to be proportional to the large eddy turnover time $T_0$. An interesting problem, not touched on here, is to find the Reynolds number dependence, if any, of the dimensionless parameters $g$, $g'$, $b_+/T_0$, and $b_-/T_0$.

\[ \text{FIG. 2. } \sigma_\omega(s) = \sqrt{\langle \omega^2(0) \rangle_L}, \text{ i.e., the standard deviation of the projected Lagrangian vorticity } \omega(0), \text{ is plotted as a function of standardized rate-of-strain eigenvalues } s/\sigma_\omega. \text{ The standard deviations of the positive and negative rate-of-strain eigenvalues } s \text{ are, respectively, } \sigma_+ = 4.63 \text{ and } \sigma_- = 6.52. \]
b_−, associated to the right and left branches of \( \sigma_\alpha(s) \), respectively, are \( b_+=a^{-1}d\sigma_\alpha(s)/ds|_{s=0}=0.26 \) and \( b_−=a^{-1}d\sigma_\alpha(s)/ds|_{s=0}=0.08 \).

In Fig. 3 we show that the expectation value ratios \( I_1^−(s)/I_1(s) \) and \( I_2^−(s)/I_2(s) \) are approximately constant for an extended range of \( s \) values, in agreement with Eqs. (2.24) and (2.25). The measured values of \( b_+, b_−, g=|I_1^−(s)/I_1(s)| \) and \( g′=I_2^−(s)/I_2(s) \) are then substituted in the logarithmic corrections written down in Eqs. (2.30) and (2.31), which are strikingly confirmed from the plots shown in Fig. 4.

The absolute ratio \( |I_1^−(s)/I_1(s)| = T_1^−(s)/T_1(s) \approx 3.84 \) is actually expected to be a number larger than unity. This follows from the well-known fact that \( s \) is most of the time negative, in other words, \( T_1(s) > T_1(s) \). We have actually verified that the domain of negative \( s \) in physical space constitutes around 75% of the total fluid volume, which is equivalent to say that the intermediate rate-of-strain eigenvalue is positively skewed [6,23].

We note, furthermore, that the estimate \( g′ \approx 15.81 \) is compatible with the relation \( g′b_− > b_− \), predicted at the end of Sec. II.

The results depicted in Figs. 3 and 4 do not depend on the specific values of the parameters \( \alpha \) and \( \beta \) introduced in Eqs. (2.16) and (2.17).

Using relations (2.6), (2.7), and (2.28)–(2.31), it follows that at \( |s|T_1(s) = 1 \), we have

\[
I_1^− = \frac{1}{\alpha(g+g′)}, \quad I_1^+ = \frac{g}{\alpha(g+g′)},
\]

while at \( |s|T_2(s) = 1 \),

\[
I_2^− = \frac{1}{\beta(g+g′)}, \quad I_1^+ = \frac{g′}{\beta(g+g′)}.
\]

Therefore, recalling that \( g \) and \( g′ \) have already been determined, the values of \( \alpha \) and \( \beta \) can be straightforwardly computed from the intercepts of the dotted lines in Fig. 4 with the vertical lines \( |s|T_1(s) = |s|T_2(s) = 1 \). Proceeding in this way, we get \( \alpha = 1.64 \) and \( \beta = 0.73 \).

Substituting the estimated values of \( \alpha, \beta, g, \) and \( g′ \) in the analytical expressions (2.28) and (2.29), we then perform consistency check for the definitions of \( I_1^−(s) \) and \( I_2^−(s) \), Eqs. (2.6) and (2.7). The results, shown in Fig. 5, are reasonable accurate, with better performance for smaller values of \( s \). For larger values \( s \), fluctuations errors come into play spoiling the comparison between analytical expressions and numerical evaluations.
happens ultimately due to the fact that the hand side of (2.18), which is assumed to yield $\sigma_\omega(s)$, is compared to the result reported in Fig. 2 (straight lines).

If, now, the numerically evaluated functions $I_1^s(s)$ and $I_2^s(s)$ take the place of $sT_1^s(s)$ and $[sT_2^s(s)]^2$, respectively, in Eq. (2.18), we would expect to recover, from an entirely alternative perspective, the empirical linear profiles depicted in Fig. 2. In fact, we find, as shown in Fig. 6, suggestive agreement for positive $s$. For negative $s$, such an evaluation of $\sigma_\omega(\omega)$ is plagued with a stronger numerical uncertainty. This happens ultimately due to the fact that $g$ and $g'$ are both larger than unity. Therefore, $I_1^s(s)$ and $I_2^s(s)$ have larger error bars than $I_1^s(s)$ and $I_2^s(s)$, respectively, which are exponentially propagated in the computation of $\sigma_\omega(\omega)$. We note, furthermore, that in Fig. 6 the range of $s/\sigma_\omega$ values is a bit smaller than the one used in Fig. 2. The range of $s/\sigma_\omega$ in Fig. 6 is actually determined by the vertical dashed lines shown (and discussed) in Fig. 3.

IV. CONCLUSIONS

The essential guideline underlying our analysis is that vorticity statistics is the ideal setting for the study of key aspects of the rate-of-strain tensor dynamics. We have devised, from the vorticity field, a suitable conditioned expectation value, $\sigma_\omega(\omega) = \sqrt{\langle \omega^2(0) \rangle}$, which is directly related to the time scales of Lagrangian strain activity. As a refinement of standard phenomenology, it turns out that two time scales, the strain persistence times $T_1(s)$ and $T_2(s)$, are necessary to accurately reproduce $\sigma_\omega(\omega)$, as determined from DNS data. The strain persistence times $T_1(s)$ and $T_2(s)$ are introduced as first- and second-order contributions within a second-order cumulant expansion, once closure and working hypotheses have been put forward.

While $|s|T_1(s)$ and $|s|T_2(s)$ vanish by construction at $s = 0$, they are both softly divergent at asymptotically large $|s|$, which happens to be a crucial ingredient in the derivation of the linear profiles of $\sigma_\omega(\omega)$, strikingly indicated in Fig. 2. The divergences of $|s|T_1(s)$ and $|s|T_2(s)$ as $s \to \pm \infty$ could shed some light on the understanding of the phenomenon of turbulent intermittency, since they suggest that strong, and hence small-scale, strain fluctuations are likely to have a non-negligible role in the statistical properties of vorticity. It is possible that the second-order truncation in the cumulant expansion (2.18) is actually a fine approximation to the full nonperturbative result, due mainly to the specific definition of the statistical ensembles $\Lambda_\omega$, which may provide a partition of the whole functional space into subspaces of Gaussian stochastic processes $s(t)$. A point in favor of the second-order cumulant expansion is the fact that the ratio between the second- and first-order contributions, $\beta |s|T_2(s)/\alpha \langle \omega(\omega) \rangle T_1(s)$, converges to $\beta(g + 1)/\alpha(g' - 1) \simeq 0.14 < 1$ as $s \to \pm \infty$.

We highlight that from a purely theoretical perspective, no considerations have been advanced to establish the form of the strain persistence times beyond the first order in $s$. However, we have found that the empirical evaluation of $\sigma_\omega(s)$ does not bring any further nonlinear corrections into scene, a fact that seems to be far from trivial [note that statistical isotropy just implies that $\langle \omega^2 \rangle = 2 \langle \omega_1^2 \rangle$, which looks like a necessary but by no means a sufficient condition for the specific observed profile of $\sigma_\omega(s)$].

There is, of course, a number of assumptions we have made throughout the paper; while they turned out to lead to reasonably good predictions of numerical results, it is clear that further experimental and numerical investigations are in order to fully support them. A natural direction of research is to check to what extent the premises and results proposed here can match the phenomenology implied by promising effective Lagrangian simulations of the velocity gradient tensor, such as the ones carried out within the Recent Fluid Deformation Closure model [24,25].
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[9] To be fair to the usual arguments, the undesirable divergence of $T(s) \sim 1/|s|$ as $s \to 0$ has not been a source of much concern, since the main focus is, in general, the role of strain in inertial range scales.
[18] Straightforward proof of the positive definiteness of $T^\pm_1(s)$: taking a look at Eqs. (2.3) and (2.6), $T^\pm_1(s) > 0$ follows from the fact that $s(t)/\lambda_1$ has always the same sign as the rate-of-strain eigenvalue $s(0) = s$.