Flow in the immediate neighbourhood of a wall: local solutions in the fully turbulent region

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Abstract

The present work shows how perturbation methods can be used to unveil important features of the turbulent boundary layer structure. The intermediate variable technique is presented and applied to problems that involve flow separation, wall roughness, compressibility, heat transfer and unsteadiness effects. Comments on shock-wave interaction, riblets and non-equilibrium flows are also made. The problem of an impinging jet is discussed to show how the present results can be extended to an apparently very distinct problem.

Keywords

Perturbation methods, intermediate variable technique, turbulent boundary layer, wall turbulence, roughness.

1 Introduction

The description of the turbulent motion in the near wall region is seriously hampered by the very complex structure that sets in. Intricate non-linear interactions between the different flow scales originate a configuration consisting on one hand, of a slow convective motion of large eddies with sizes comparable to the boundary layer thickness and much larger than the energy containing eddies and, on the other hand, of low-speed streaks in the near wall region. This picture has been identified by many authors with a double structure.

As it turns out, turbulence production, kinetic energy and dissipation reach their maxima within the very thin viscous layer, in the immediate neighbourhood of the wall. The implications for computational methods are immediate: (i) extremely fine meshes are required to capture the very strong near wall gradients of flow quantities and (ii) complex turbulence models are needed to represent all relevant flow features. Both difficulties pay a heavy load on the required computational time.

In fact, for some time authors sustained that in the low-Reynolds-number wall region, turbulence closure could be achieved through consideration of viscosity dependent coefficients. Modifications were largely empirical and so constructed as to make sure that the flow properties in the fully turbulent region were completely recovered. The apparent universality of low-Reynolds-number models, however, suffered from the inherent inability of damping functions to account for some physical effects including the isotropic viscous damping of all turbulent fluctuations and the non-isotropic, non-viscous suppression of the vertical velocity fluctuations due to wall non-penetrability (Popovac and Hanjalic, 2007). Damping functions also introduce non-linearity and numerical stiffness, and require very fine computational meshes.

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Strategies to avoid the need for integration of the governing equations up to the wall were developed in the mid-sixties. The fundamental idea was to patch the numerical two-dimensional outer region solution to a one-dimensional analytical near-wall solution that embedded all low-Reynolds-number effects. This "inner" solutions were expressed in terms of the dimensionless variables based on the fluid viscosity and local wall stress. For instance, Spalding (1967) advanced some elaborate expressions to account for pressure gradient and wall transpiration effects. However, most of the existing software still appeal to very simple wall-functions.

The purpose of the present work is to review the asymptotic arguments that lay ground for the establishment of a double-layered structure in terms of the intermediate variable tech¬nique of Kaplun (1967). A discussion on changes of the flow structure due to the proximity of a separation point is also presented. The work shows how elaborated expressions can be derived to represent the flow in the fully turbulent region so as to account for many distinct effects.

The works of Cruz and Silva Freire (1998) and Loureiro and Silva Freire (2011a) are the main references for the fundamental results on the intermediate variable technique. The usefulness of the asymptotic analysis is illustrated with the derivation of local solutions for several problems of practical interest. Important physical effects of disturbances on a boundary layer flow including separation, wall roughness, compressibility, heat transfer, fluid transpiration, change in wall surface and shock-wave interaction can be modelled through perturbation techniques. In the following sections, local solutions are developed in the scope all these problems.

2 Perturbation analysis: morphological structure

To find the asymptotic structure of the turbulent boundary layer of a Newtonian fluid, con-sider the system of Eqs. (1) and (2),

$$\partial_i u_i = 0, \tag{1}$$

$$u_i \partial_j u_i = -\rho^{-1}\partial_i \rho - \epsilon_2 \partial_j \left(\overline{u_j' u_i'}\right) + \epsilon^2 \hat{\epsilon}\, \partial^2 \mathrm{u_i}, \tag{2}$$

where the notation is classical. Thus, in a two-dimensional flow, $(x_1 x_2) = (x, y)$ stands for a Cartesian co-ordinate system, $(u_i, u_i) = (u, v)$ for the velocities, p for pressure. The dashes are used to indicate a fluctuating quantity. In the fluctuation term, an overbar is used to indicate a time-average.

All mean variables are referred to the free-stream mean velocity, u_e, and to the characteristic length $l = (\rho u_e^2/(\partial_x p)_w)$, ($w$ = wall conditions). The leading order velocity fluctuations are considered to be of the order of the friction velocity $u_* (= \sqrt{\tau_w/\rho})$ so that $\epsilon = u_*/u_e$. The second small parameter is defined as $\hat{\epsilon} = 1/(\epsilon^2 R)$, where $R\ (= u_e l/\nu)$ is the Reynolds number.

The asymptotic structure of the flow will be determined through the single limit concept of Kaplun. The fundamental notions on perturbation methods to be used henceforth were laid down by Kaplun (1967), Lagerstrom and Casten (1972) and Lagerstrom (1988) in extensive texts. More recent contributions can be found in Cruz and Silva Freire (1998) and Loureiro and Silva Freire (2011a). In particular, the article of Loureiro and Silva Freire (2011a) discusses in detail most of the relevant definitions and results.

To keep the present work within a permissible level of understanding, only the essential information will be repeated here. The topology on the collection of order classes introduced by Meyer (1967) is used. For positive, continuous functions of a single variable f defined on (0, 1], ord η denotes the class of equivalence introduced in Meyer.

Definition (Lagerstrom (1988)). We say that $f(x,\epsilon)$ is an approximation to $g(x,\epsilon)$ uniformly valid to order $\delta(\epsilon)$ in a convex set D (f is a δ-approximation to g), if

$$lim(f(x,\epsilon) - g(x,\epsilon))/\delta(\epsilon) = 0, \quad \epsilon \to 0$$
uniformly for x in D (3)

Consider

$$x_\eta = x/\eta(\epsilon),\ \ G(x_\eta; \epsilon) = F(x; \epsilon) \tag{4}$$

with $\eta(f)$ defined in Ξ (= space of all positive continuous functions on (0,1]).

Definition (of Kaplun limit)(Meyer (1967)). If the function $G((x_\eta; +0) = lim\ G(x_\eta;\ \epsilon),\ \epsilon \longrightarrow 0$, exists uniformly on $\{x_\eta / |x_\eta| > 0$; then we define $lim_\eta\ F(x;\epsilon) = G(x_\eta; +0)$,

The definition of η-limit of a function and of domains of validity were given an analogous concept for equations by Lagerstrom and Casten (1972). They made the following definitions.

Definition (Lagerstrom and Casten (1972)). If E is an equation and $lim_{\eta 1} E = E_1;\ lim_{\eta 2} E = E_2$ and also $lim_{\eta 2} E_1 = E_2$, we say that E_1 contains E_2 (relative to E).

Definition (Lagerstrom and Casten (1972)). The formal domain of validity of an equation F, relative to the "full" equation E, is the ord η such that $lim_\eta E$ is either F or an equation contained in F.

The above definitions naturally imply the existence of distinguished equations, obtained from specific choices of η. These equations are, in the sense of Kaplun (1967), "rich" equations. A more elaborate statement is given by

Definition. An equation P that contains other limit equations but is not contained by any other is said to be a principal equation.

An equation which is not principal is said to be an intermediate equation.

The previous definitions are complemented by the following statement,

Principle (Kaplun (1967)). If y is a solution of an equation E and E^* is an approximate equation, then there exists a solution y^* of E^* whose actual domain of validity (as an approximation to y) includes the formal domain of validity of E^* (as an approximation to E).

To analyze the turbulent boundary layer, make

$$\hat{y} = y_\eta = y/\eta(\epsilon), \quad \hat{u}_i(x, y_\eta) = (u_i\, x, y) \tag{5}$$

Upon substitution of Eq. (5) into Eqs.(1) and (2) and depending on the order class of η we then find the following formal limits: continuity equation:

$$\text{ord}(\hat{v}(x, y_\eta)) = ord(\eta \hat{u}(x, y)) \tag{6}$$

x-momentum equation:

$$ord\eta = ord1 : \ \hat{u}\partial_x \hat{u} + \hat{v}\partial_{y_\eta}\hat{u} + \partial_x \hat{p} = 0 \tag{7}$$

$$ord\epsilon^2 < ord\eta < ord1 : \ \hat{u}\partial_x \hat{u} + \hat{v}\partial_{y_\eta}\hat{u} + \partial_x \hat{p} = 0 \tag{8}$$

$$ord\epsilon^2 = ord\eta : \ \hat{u}\partial_x \hat{u} + \hat{v}\partial_{y_\eta}\hat{u} + \partial_x \hat{p} = \partial_{y_\eta}\overline{u'v'} \tag{9}$$

$$ord(\epsilon\hat{\epsilon}) < ord\eta < ord\epsilon^2 = ord\eta \ : \partial_{y_\eta}\overline{u'v'} = 0 \tag{10}$$

$$ord(\epsilon\hat{\epsilon} = ord\eta : \ -: \partial_{y_\eta}\overline{u'v'} + \partial^2_{y_\eta}\hat{u} = 0 \tag{11}$$

$$ord\eta < ord(\epsilon\hat{\epsilon}) : \partial^2_{y_\eta}\hat{u} = 0. \tag{12}$$

y-momentum equation:

$$ord\eta = ord1 : \ \hat{u}\partial_x \hat{v} + \hat{v}\partial_{y_\eta}\hat{v} + \partial_{y_\eta}\hat{p} = 0 \tag{13}$$

$$ord\eta = ord1 : \ \partial_{y_\eta}\hat{p} = 0. \tag{14}$$

The above results show that the turbulent boundary layer fluid exhibits a two-deck structure defined by the principal equations, Eqs. (9) and (11). The viscosity of the fluid defines the thickness of the viscous region through $\epsilon\hat{\epsilon} = 1/(\epsilon^2 R)$ The turbulence dominated region is defined by $ord(\epsilon\hat{\epsilon}) < ord(\varsigma) < ord(\epsilon^2)$).

Therefore, the principal equations to the turbulent boundary layer problem are Eqs. (9), (11) and (13). The relevant scales ϵ^2 and $\epsilon\hat{\epsilon}$ coincide with the scales proposed by Sychev and Sychev (1987) for the description of their two internal layers. These authors also consider a third layer. However, in the interpretation of Kaplun limits, this is not necessary for only redundant information is conveyed. An important point to be raised here is the nature of the principal equation in y-direction. To solve the boundary layer equations one needs to consider Eq. (13), instead of the Prandtl formulation $\partial_y p = 0$. By doing this, the boundary layer approximation becomes a self-contained theory in the sense that any type of viscousinviscid interactive process becomes unnecessary.

In respect to the determination of Eq. (11), please, refer to further arguments presented in Loureiro and Silva Freire (2011a). The matching process that involves the inner and outer solutions (Eqs. (9) and (11)) presents a peculiar difficulty, sometimes referred to in literature as 'generation gap' (Mellor, 1972). When this happens, an inspection of formally higher order terms leads to 'switchback' and to a change in the leading order of the inner solution.

As the flow approaches a separation point, however, the structure depicted by Eqs. (6) to (14) breaks down since $u_* \longrightarrow 0$. To account for the flow behaviour, we must then consider Kaplun limits in x-direction.

Define

$$\begin{aligned} \hat{x} &= x_\Delta = x/\Delta(\epsilon), \\ \hat{y} &= y_\eta = y/\eta(\epsilon), \\ \hat{u}_i(x_\Delta, y_\eta) &= u_i(x, y). \end{aligned} \tag{15}$$

with $\Delta(\epsilon)$ and $\eta(\epsilon)$ defined on Ξ.

The idea is to approach the separation point by taking simultaneously the η-and Δ-limits at a fixed rate $\zeta = \Delta/\eta = ord(1)$.

Close to a separation point, Loureiro and Silva Freire (2011a) show that a different near wall scaling must be used with a small parameter ϵ dependent on the local pressure gradient ($\epsilon = u_{pv}/u_e, u_{pv} = ((\nu/\rho)\partial_x p)^{1/3}$. Under this condition, the following result applies, $ord(\epsilon^2) = ord(1/\epsilon R)$.

The resulting flow structure is then given by continuity equation:

$$ord\left(\hat{v}(x, y_\eta)\right) = ord(\hat{u}(x, y_\eta)). \tag{16}$$

x-momentum equation:

$$ord\Delta = ord1 : \ \hat{u}\partial_{x_\Delta}\hat{u} + \hat{v}\partial_{y_\eta}\hat{u} + \partial_{x_\Delta}\hat{p} = 0. \tag{17}$$

$$ord\epsilon^2 < ord\Delta < ord1 : \hat{u}\partial_{x_\Delta}\hat{u} + \hat{v}\partial_{y_\eta}\hat{u} + \partial_{x_\Delta}\hat{p} = 0. \tag{18}$$

$$\begin{aligned} ord^2 = ord\Delta : \ &\hat{u}\partial_{x_\Delta}\hat{u} + \hat{v}\partial_{y_\eta}\hat{u} + \partial_{x_\Delta}\hat{p} = \\ &-\partial_{x_\Delta}\overline{u'^2} - \partial_{y_\eta}\overline{u'v'} + \partial^2_{x_\Delta}\hat{u} + \partial^2_{y_\eta}\hat{u}. \end{aligned} \tag{19}$$

$$ord\Delta < ord\epsilon^2 : \ \partial^2_{x_\Delta}\hat{u} + \partial^2_{y_\eta}\hat{u} = 0. \tag{20}$$

y-momentum equation:

$$ord\Delta = ord1 : \ \hat{u}\partial_{x_\Delta}\hat{v} + \hat{v}\partial_{y_\eta}\hat{v} + \partial_{y_\eta}\hat{p} = 0. \tag{21}$$

$$ord\epsilon^2 < ord\Delta < ord1 : \hat{u}\partial_{x_\Delta}\hat{v} + \hat{v}\partial_{y_\eta}\hat{v} + \partial_{y_\eta}\hat{p} = 0. \tag{22}$$

$$ord\epsilon^2 = ord\Delta : \hat{u}\partial_{x_\Delta}\hat{v} + \hat{v}\partial_{y_\eta}\hat{v} + \partial_{y_\eta}\hat{p} = -\partial_{x_\Delta}\overline{u'v} - \partial_{y_\eta}\overline{u'^2} + \partial^2_{x_\Delta}\hat{v} + \partial^2_{y_\eta}\hat{v}. \quad (23)$$

$$ord\Delta < ord\epsilon^2 : \partial^2_{x_\Delta}\hat{v} + \partial^2_{y_\eta}\hat{v} = 0. \quad (24)$$

The principal equations are Eqs. (19) and (23). They show that near to a separation point the two principal equations, Eqs. (9) and (11), merge giving rise to a new structure dominated basically by two regions: a wake region ($ord(\eta)$, $ord(\Delta) > \epsilon^2$) and a viscous region ($ord(\eta)$, $ord(\Delta) < \epsilon^2$). The disappearance of the region dominated solely by the turbulence effects is noted. The principal equations recover the full Reynolds averaged Navier-Stokes equations.

The system of Eqs (17) to (24) indicates that the pressure gradient effects become leading order effects for orders higher than $ord(\epsilon^2) = ord(\Delta)$. Thus, at about $ord(x/l) = ord(\Delta) = ord(\epsilon)$ we should have $ord(u_* = ord(u_{pv})$, so that these terms furnish first order corrections to the mean velocity profile.

3 Near wall approximate solutions for attached and separated flows over smooth surfaces

The present section illustrates how the intermediate equations derived above can be used to find some very simple approximated flow solutions.

The two-layered model shows that there exists a fully turbulent flow region where the x-motion equation reduces to

$$\partial_y(-p\overline{u'v'}). \quad (25)$$

A simple integration of the above equation implies that $ord(u') = ord(v') = ord(u_*)$, where we have considered the velocity fluctuations to be of the same order.

The analysis proceeds by taking as a closure assumption the mixing-length theory. A further equation integration yields the classical law of the wall for a smooth surface

$$u^+ = \varkappa^{-1}\ln y^+ + A, \quad (26)$$

where $u^+ = u/u_*, y^+ = y/(\nu/u_*), \varkappa = 0.4, A = 5.0$.

For laminar flow, at a point of zero skin-friction the velocity profile must follow a y²-profile at the wall. For turbulent flow, the fact that the local leading order equations must be dominated by viscous and pressure gradient effects implies immediately that this result remains valid.

In fact, in the viscous region the local governing equation can be written as:

$$\nu\partial_{yy}u = \rho^{-1}\partial_x p. \quad (27)$$

Two successive integrations of Eq. (27) and the fact that $\tau_w = 0$, give

$$u^+ = (1/2)y^{+2}. \quad (28)$$

with $u^+ = u/u_{pv}, y^+ \, y/(\nu/u_{pv}), u_{pv} = ((\nu/p)\partial_x p)^{1/3}$.

In Eq. (28) the term $\partial_x p$ must be evaluated at $y = 0$. Hence, wall similarity solutions cannot be expressed in terms of the external pressure gradient.

For the turbulence dominated region, Stratford (1959) wrote

$$\partial_y\tau_t = \partial_x p. \quad (29)$$

Two successive integrations of Eq. (29) together with the mixing length hypothesis and, again, the fact that at a separation point τ_w, give

$$u^+ = (2\varkappa^{-1})y^{1/2}, \quad (30)$$

with u^+ and y^+ defined as in Eq. (28).

To find his solution Stratford used the condition $y = 0$, $u = 0$. Strictly speaking, this condition should not have been used since Goldstein's y^2-expression is the solution that is valid at the wall. Stratford also incorporated an empirical factor β (= 0.66) - to Eq. (30) to correct pressure rise effects on $\varkappa$.

Thus, we may conclude that, at a separation point, $ord(u') = ord(v') = ord(u_{pv})$.

To find a solution over the entire viscous sublayer, for attached as well as detached flow, Eqs. (25) and (27) must be combined to give

$$\nu\partial_{yy}u + \partial_y\left(-\rho\overline{u'v'}\right) = \rho^{-1}\partial_x p. \quad (31)$$

The global solution of Eq. (31) should also reduce, under the relevant limiting processes, to the local approximate solutions.

A double integration of Eq. (31) in the fully turbulent region $\left(\mu\partial^2_y u \approx 0\right)$ furnishes (see, e.g., Cruz and Silva Freire (1998))

$$u = 2\varkappa^{-1}\sqrt{\Delta_w} + \varkappa^{-1}u_*\ln\left(\left(\sqrt{\Delta_w} - u_*\right)/\left(\sqrt{\Delta_w} + u_*\right)\right) + C, \quad (32)$$

with $\Delta_w = \rho^{-1}\tau_w + (\rho^{-1}\partial_x p)y$.

Equation (32) must be viewed with much discretion for depending on the relative values of τ_w and $(\partial_x p)y$ the discriminant Δ_w might become negative, thus rendering the solution undetermined. Furthermore, the argument of the logarithmic term cannot become negative. In Cruz and Silva Freire (1998), three different cases have been identified and explicitly quoted.

In general, however, Eq. (32) can be seen as a generalization of the classical law of the wall for separating flows. In the limiting case $(\partial_x p)y << \tau_w$, Eq. (32) reduces to the logarithmic expression

$$u^+ = \varkappa^{-1}\ln y^+ + b_m, \tag{33}$$

$$b_m = 2\varkappa^{-1} + \varkappa^{-1}\ln\left(\left(u_{pv}^3/4u_*^3\right)e^{\varkappa C}\right). \tag{34}$$

Near a point of separation Stratford's solution is recovered.

In principle, Eq. (32) can be used indistinctly in all flow regions - including regions of reversed flow - provided the domain of validity of its discriminant is respected and appropriate integration constants are determined. Equation (32) cannot be written in terms of the similarity variables u_* and upv for in situations where any of these two parameters approaches zero, a singularity occurs. Thus, it will be kept in its present form.

Other different treatments of the lower boundary condition can be found in the literature to model separating flows. For two-dimensional, smooth, steep hills, Loureiro et al. (2007a) have investigated the performance of the formulations introduced by Mellor (1966), by Nakayama and Koyama (1984) and by Cruz and Silva Freire (2002). A much more detailed analysis of separated flow over a steep, smooth hill can be found in Loureiro et al. (2007b).

4 Near wall approximate solutions for attached and separated flows over rough surfaces

The effects of roughness on a boundary layer can be dramatic. Provided the characteristic size of the roughness elements are large enough, a regime can be established where the flow is turbulent right down to the wall (fully rough flow). One important consequence is that the viscous sublayer is completely removed so that the linear and Goldstein's solutions do not apply anymore. The roughness also distorts the logarithmic profile acting as if the entire flow is displaced downwards.

The manner in which the logarithmic law is expressed to describe flow over a rough surface depends on the field of application. In meteorology, the common practice is to write

$$u^+ = \varkappa^{-1}\ln\left((y-d)/y_0\right), \tag{35}$$

where y is the distance above the actual ground surface.

The specification of the lower boundary condition on rough walls depends thus on two unknown parameters: the aerodynamic surface roughness, y_0, and the displacement height, d. Many works have attempted to relate the magnitude of d and y_0 to geometric properties of the surface. Garratt (1992) mentions that the simple relation $d/h_c = 2/3$ (h_c= height of canopy) seems to offer good results for many of the natural vegetation of interest. However, since d is known to depend strongly on the way roughness elements are packed together, much discretion must be considered in using this relation. Garratt (1992) also mentions that many texts suggest considering $y_0/h_c = 0.1$. Typical natural surfaces satisfy $0.02 < y_0/h_c < 0.2$

The arguments that lead to Stratford's law are based on the fundamental hypothesis that near a separation point a fully turbulent region can be identified in the flow. This consideration remains valid for flow over a rough surface. The direct implication is that the procedure that resulted in the derivation of Eq. (30) can be repeated for flow over rough surface but with $y^+ = (y-d)/y_0$ and $u_{pv} = ((y_0/\rho)\partial_x p))^{1/2}$. The integration constant must also be determined so as to correctly account for the roughness effects.

The derivation of Eq. (32) has disregarded any detail of the wall roughness. This equation is, in fact, supposed to be valid not in the region adjacent to the wall where the complicated flow around the individual roughness elements is apparent, but, instead, in a region where the flow statistics are spatially homogeneous. Hence, inasmuch as for the classical law of the wall, the characteristics of the rough surface must enter the problem through the integration constant C. In addition, the coordinate system must be displaced by d. The immediate conclusion is that Eq. (32) can be used to model separating flow over a rough surface provided d and C are adequately modeled.

Parameter C is a general function of τ_w, $\partial_x p$ and y_0 that must be determined by a con-sistent analysis of experimental data. However, an estimate of its functional form might be obtained by considering the limiting behaviour of Eq. (32) as $\tau_w << (\partial_x p)y$. The resulting expression is

$$C = \varkappa^{-1}u_*[\ln(4u_*^2/((\rho^{-1}\partial_x p)y_0)) - 2] \tag{36}$$

This parametrization scheme was first presented in Loureiro et al. (2008). A detailed comparison with experimental data has been presented in Loureiro et al. (2009) and Loureiro and Silva Freire (2009).

5 Near wall approximate solutions for flows with wall transpiration

For flow subject to wall transpiration, the asymptotic structure of the turbulent boundary layer does not change. Since the wall injection enters the problem as a regular perturbation parameter (Silva Freire, 1988a), the flow structure remains double-tied.

In fact, the result of the injection or suction of fluid into an oncoming flow is to modify the velocity distribution throughout the boundary layer so that drag is either reduced or in¬creased. Any expression advanced with the purpose of determining the friction coefficient should therefore reflect this.

Regarding the inner layer equations of motion, the effects of flow injection can be account for through consideration of contributions by the inertia term, by equation

$$v_w \partial_y u = \nu \partial_{yy} u + \partial_y\left(-\rho \overline{u'v'}\right) \tag{37}$$

In Silva Freire (1988a) the matched asymptotic expansions method was applied to the equations of motion to find a law of the wall in which the additive parameter A varied with transpiration. The resulting expression is

$$u^+ = \varkappa^{-1}\ln(y^+) + A + \Pi\varkappa^{-1}W(y\delta^- + v_w^+\left((2\varkappa)^{-1}\ln(y^+) + 2^{-1}A\right)^2 + \widetilde{\Pi}\varkappa^{-1}W(y\delta^{-1}) \tag{38}$$

with $u^+ = uu_*^{-1}, y^+ = yu_*\nu^{-1}, v_w^+ = v_w u_*^{-1}, v_w$ = normal velocity at the wall and A is given by:

$$A = 5 - 512(v_w U^{-1}) \tag{39}$$

and parameters Π and $\widetilde{\Pi}$ and function W are related to the universal wake function.

The above equations are valid for incompressible, isothermal flows over smooth surfaces. They have been derived for external flows, but can be easily specialized to describe pipe flows. This has been made in Loureiro and Silva Freire (2011b), where a resistance law is proposed for pipe flows with wall transpiration.

Compressibility effects can be accounted for provided the concept of generalized velocity is summoned.

Van Driest (1951) solved the equations of motion for a compressible, adiabatic flow to find

$$\frac{u_\infty}{\xi u_*}\left(\arcsin\xi - \arcsin\frac{\xi u}{u_\infty}\right) = -\frac{1}{\varkappa}\ln\frac{y}{\delta} \tag{40}$$

where

$$\xi^2 = (\gamma - 1)\, M^2/(2 + (\gamma - 1)M^2), \tag{41}$$

with M = Mach number.

Comparison of Eq.(40) with the log-law of the wall shows that the compressibility effects can be accounted for provided the incompressible velocity profile is replaced by the generalized velocity, u_α, defined by

$$u_\alpha = \frac{u_\infty}{\xi}\arcsin\frac{\xi u}{u_\infty}. \tag{42}$$

The above equation is normally referred to as the Van Driest transformation. In the limit as $M \rightarrow 0$, Eq.(40) reduces to the incompressible flow case.

The extension of Eq. (38) to the compressible case with application of transformation Eq. (42) has been discussed in Silva Freire (1988b).

6 Near wall approximate solutions for flows with wall heat transfer

Consider the problem of a given incompressible fluid flowing over a smooth, heated surface under a steady state condition.

In the following analysis, for the sake of completeness, the mean velocity and temperature fields are discussed together. The purpose is to show how simple analogies can be drawn between the two transfer processes.

The governing equations are:

Continuity:

$$\partial_x u + \partial_y v = 0, \tag{43}$$

x-Momentum:

$$\rho u \partial_x u + \rho v \partial_y u = \mu \partial_{yy} u - \rho \partial_{yy} u - \rho \partial_y \overline{u'v'}, \tag{44}$$

Energy:

$$\rho c_p u \partial_x T + \rho c_p y \partial_y T = K \partial_{yy} T - \rho c_p \partial_y \overline{v't'}, \tag{45}$$

where the notation is classical and the boundary layer hypothesis applies.

These equations must be solved under appropriate boundary conditions at the wall. For the velocity field, the no-slip condition and the permeability condition can be used. For the temperature field, a number of different possible boundary conditions can be specified. Basically, one can prescribe the wall temperature, the wall heat flux or a combination of these two.

Consider next that, as shown before, the turbulent boundary layer has a two-layered structure, and that, furthermore, in one of the existing layers the turbulence effects dominate. Since the general mathematical structure of the temperature problem remains the same as that for the velocity problem,

Thus, in this layer, the governing equations reduce to:

x-Momentum:

$$\partial_y \overline{u'v'} = 0, \tag{46}$$

Energy:

$$\partial_y \overline{v't'}. \tag{47}$$

So that the above equations can be solved, a relation has to be established between the mean and the turbulent quantities. The simplest way of doing this is to invoke the concepts of eddy diffusivities for momentum and heat, together with the mixing-length hypothesis. This results in the following algebraic equations for the turbulent quantities

$$-\partial_y \overline{u'v'} = \partial_y[\nu_t \partial_y u] = \partial_y\left[l^2 (\partial_y u)^2\right] = 0, \tag{48}$$

$$-\partial_y \overline{v't'} = \partial_y[a_t \partial_y u] = \partial_y[(l\partial_y u)(l_t \partial_y T)] = 0. \tag{49}$$

where ν_t and a_t denote the eddy diffusivities for momentum and heat.

We further incorporate into our analysis two extra hypotheses:

1. von Karman's hypothesis that the mixing–length can be considered proportional to the wall distance, i.e. $l_t = \varkappa y$ and $l_t = \varkappa_t y$, where $\varkappa$ and $\varkappa_t$ are constants.

2. Prandtl's hypothesis that in the near wall region the total shear stress and the heat flux are constant.

Thus, upon a simple integration, it results that in the fully turbulent region the local solu-tions are given by:

$$u^+ = \varkappa^{-1} \ln y + A, \tag{50}$$

and

$$T^+ = \varkappa_t^{-1} \ln y + B \tag{51}$$

where $u_+ = u/u_*$, $u_* = \sqrt{(\tau_w/\rho)}$ and $T^+ = (T_w - T)/t_*$, $t_* = q_w/(\rho c_p u_*)$.

The implication of Eqs. (50) (previously shown as Eq. (26)) and (51) is that, provided $\varkappa$ and $\varkappa_t$ are known, the skin-friction coefficient and the heat-transfer coefficient can be evaluated respectively from the inclination of semi-log graphs of distance from the wall versus velocity and distance from the wall versus temperature.

If a turbulent Prandtl number is defined, it follows that

$$Pr_t = \frac{\nu_t}{a_t} = \frac{\varkappa}{\varkappa_t}. \tag{52}$$

A common sense in literature is that Pr_t varies across the boundary layer in a way that depends on both the molecular properties of the fluid and the flow field. In the logarithmic region, however, several authors have shown that Pr_t is approximately 0.9 which results in a value of 0.44 for $\varkappa_t$.

7 Transient convection in turbulent boundary layers over smooth surfaces

Consider now the transient convection in turbulent boundary layers over smooth, flat surfaces. The velocity field remains unaltered so that the velocity local solution in the fully turbulent region can still be approximated by the logarithmic equation, Eq. (50).

The thermal problem, however, suffers an important modification since the surface boundary conditions have to change to accommodate a time varying imposed heat flux.

Thus, it results that the energy governing equation become

$$\partial_t T = -\partial_y \overline{v't'}. \tag{53}$$

In view of the results of Section 2, the above equation can be re-written as

$$\partial_t T = \partial_y (u_* \varkappa_t y \partial_y T). \tag{54}$$

To find a solution, consider

$$T(y,t) = F(t)G(y). \tag{55}$$

Then upon substitution of Eq. (55) onto Eq. (54) it follows that

$$\frac{F'(t)}{F(t)} = u_\tau \varkappa_t \left[\frac{G'(y) + yG''(y)}{G(y)}\right] \tag{56}$$

So that a solution is sought from equations

$$\frac{F'(t)}{F(t)} = -\sigma, \tag{57}$$

$$G'(y) + yG''(y) + \frac{\sigma}{u_\tau \varkappa_t} G(y) = 0, \tag{58}$$

where the sign of σ was chosen so as to ensure that the temperature will decay in time.

The solution of Eq. (57) is

$$F(t) = Je^{\sigma t}. \tag{59}$$

To solve Eq. (58) consider the decaying time to be long enough so that ($\epsilon = \sigma / \varkappa u\tau$) can be considered a small parameter. Then, search for a solution of the form

$$G(z) = G_0(y) + \epsilon G_1(y). \tag{60}$$

The substitution of Eq. (60) onto Eq. (58) and the collection of the terms of the same order yields

$$G_0'(y) + yG_0''(y) = 0, \tag{61}$$

$$G_1'(y) + yG_1''(y) + G_0(y) = 0, \tag{62}$$

whose solutions are

$$G_0(y) = C \ln y + D, \tag{63}$$

$$G_1(y) = E \ln y + R\, y \ln y + S\, y + Q, \tag{64}$$

with $R = C$ and $2C + D - \underline{S} = 0$. Thus, the fully turbulent approximate solution is given by

$$T(y,t) = Je^{-\sigma t}[(C \ln y + D) + (\sigma/\varkappa_t u_\tau)) (E \ln y + R\, y \ln y + S\, y + Q)], \tag{65}$$

where all constants must be determined experimentally.

8 Transient convection in turbulent boundary layers over rough, flat surfaces

If all above results are to be extended to flows over rough surfaces of the types 'K' or 'D', the classical two–layered structure of the boundary layer will have to be abandoned.

We know that for flows over 'K' or 'D' rough surfaces the viscous region is completely destroyed by the protuberances at the wall. Under this condition, the fully turbulent region just described above has to suffer some adjustments so as to yield a good description of the velocity and the temperature fields. Other authors have shown that a universal expression can be written for the wall region provided the origin for measuring the velocity profile is set at some distance below the crest of the roughness elements. This displacement in origin is normally referred to in literature as the error in origin, ε.

Thus, for any kind of rough surface, it is possible to write

$$u^+ = \frac{1}{\varkappa} \ln \left[\frac{(y_T + \varepsilon) u_*}{\nu}\right] + A - \frac{\Delta u}{u_*}, \tag{66}$$

where,

$$\frac{\Delta u}{u_*} = \frac{1}{\varkappa} \ln \left[\frac{\varepsilon u_*}{\nu}\right] + C_i \tag{67}$$

the subscript T is used to indicate that the origin is to be taken at the top of the protuberances (and this must not be confused with the subscript t used also to indicate temperature), $\varkappa = 0.4$, $A = 5.0$ and C_i, $i = K, D$; is a parameter characteristic of the roughness.

Equations (66) and (67), although of a universal character, have the inconvenience of needing two unknown parameters for their definition, the skin-friction velocity, u_*, and the error in origin, ε. A chief concern of many works on the subject is, hence, to characterize these two parameters.

For an experimentalist, however, these equations are very useful for they provide a graphical method for the determination of the skin-friction coefficient.

To extend Eq. (65) to turbulent flows over rough surfaces we will draw a direct analogy with Eq. (66).

For flows over rough surfaces, we have just seen that the characteristic length scale for the near wall region must be the displacement in origin. Indeed, in this situation, the viscosity becomes irrelevant for the determination of the inner wall scale because the stress is transmitted by pressure forces in the wakes formed by the tops of the roughness elements. It is also clear that, if the roughness elements penetrate well into the fully turbulent region, then the displaced origin for both the velocity and temperature profiles will always be located in the overlap fully turbulent region.

The similarity in transfer processes for turbulent flows then suggests that

$$T(y,t) = Je^{-\sigma t}[(C \ln y^+ + D) + (\sigma/\varkappa_t u_\tau)) (E \ln y^+ + R y^+ \ln y^+ + S y^+ + Q)], \tag{68}$$

where $y^+ = (y_T + \varepsilon_t)\, u_*/\nu$ and the parameters to be determined may now be a function of the roughness.

In principle, the error in origin for the temperature, ε_t, should be time dependent.

Eq. (68), however, provides a good means to measure the heat flux at the wall. Provided we can evaluate the error in origin through one of the classical techniques, the slope of the temperature profile plotted in a semi-log graph will furnish the friction temperature and, thus, the heat transfer coefficient.

9 Near wall approximate solutions for flows with wall heat transfer and wall transpiration

To study the effects of transpiration on the thermal boundary layer we again split the flow region into distinct parts where certain dominant effects can be used to derive simplified equations. The formulation of the transpiration problem basically differs from the solid surface problem in the sense that the inertia effects near the wall can no longer be neglected (Eq. 37). Therefore, for the near-wall dominated part of the flow, the approximate energy equation becomes

$$\rho c_p v_w \partial_y T = K a_{yy} T + \rho c_p \partial_y \left(-\overline{v' t'}\right) \tag{69}$$

In the fully turbulent region, the mixing-length hypothesis can be considered and the molecular diffusivity neglected. Hence a double integration of Eq. (69) gives

$$\phi = \frac{1}{\varkappa_t} \ln \frac{y^+}{y_b^+} \tag{70}$$

where

$$\phi = \frac{\varkappa_m}{\varkappa_t} \frac{2}{v_w^+} \left[\left(\frac{v_w^+ t^+ + 1}{v_w^+ t_b^+ + 1} \right)^{\varkappa_t / 2\varkappa_m} \left[\frac{v_w^+}{2} \left(\frac{1}{\varkappa_m} \ln \frac{y_a^+}{y_b^+ P_r} \right) + \sqrt{1 + v_w^+ u_a^+} \right] - \sqrt{1 + v_w^+ u_a^+} \right] \tag{71}$$

Here, $v_w^+ = v_w u_{-1}^*$ the pair (y_b^+, u_b^+) is a constant of integration and parameters $\varkappa_m$ and $\varkappa_t$ are characteristic of the turbulence modelling. The above equation can be used to determine a Stanton number equation. This has been discussed in Faraco-Medeiros and Silva Freire (1992).

The above equation can be extended to compressible flow (Silva Freire et al. (1995)). The analysis is quite evolving and for this reason will not be repeated here. The approach resorted to asymptotic techniques to divide flow region into distinct parts so that the dominant effects could be used to derive simplified sets of equations. The resulting equations for the near wall region were then integrated, yielding analytical solutions for the main flow parameters.

From these solutions, the influence of the dissipation terms and of the injection velocity was clearly seen. For the solid surface case, it was shown that the dissipation contributes to the leading order solution with a bilogarithmic term. For flows with transpiration, however, the dissipation also contributes with a trilogarithmic higher order correction. Also, it emerged from the analysis that the dissipation effects become important only as $E = 0(u_*)$, E = Eckert number.

The near wall solutions were extended to the defect region by adding Coles' function to their logarithmic term. With arguments similar to those of Faraco-Medeiros and Silva Freire (1992), a Stanton number equation was developed. All predictions are compared with the experimental data of other authors. Both parameters in the law of the wall were shown to vary with Mach number, whereas only one of them, the linear coefficient of the straight part of the velocity and temperature profiles, was shown to vary with the injection rate. No dependence of these parameters on Eckert number could be determined due to the scatter in the experimental data.

10 Wall-layer velocity profile for an impinging jet

The methodology set out in the previous sections can be used to tackle some quite different problem. Next, we show how that can be made for a impinging jet.

For an impinging jet, Özdemir and Whitelaw (1992) have shown that a Weibull distribution represents well some of the global features of the profile, such as the position of the maximum and of the outer inflection points, but is not an adequate approximation for the near wall region. For this region, they showed that semi-logarithmic relation can be used to model the inner equilibrium layer, so that one can write

$$u^+ = \varkappa^{-1} \ln y^+ + A(U_M / u^*) \tag{72}$$

The main contribution of Özdemir and Whitelaw (1992) was to show that, for the impinging jet, the inner layer appears to constitute a considerable part of the inner boundary layer, and, if the outer edge of the equilibrium layer is attached to the point of maximum velocity, which is very close to the wall, then, this maximum, U_M, should be an appropriate velocity scale. The conclusion, therefore, is that parameter A is not invariant but changes with a deviation function.

Wygnanski et al. (1992) remarked that, for a turbulent wall jet, the velocity profile cannot be universally represented in wall coordinates, as it can in the boundary layer. That is due to large variations in the additive constant in the law of the wall. In fact, depending on the jet Reynolds number, logarithmic fits can be found to their data in regions defined by specific limits. These fitted straight lines have levels varying from 5.5 to 9.5. The existence of a well defined logarithmic region is particularly important for the determination of the skin-friction. Wygnanski et al. further remark that in previous experiments the skin-friction was either directly assessed through floating drag balances or indirectly by wall heat transfer devices or by impact probes like Stanton probes or Preston tubes. Since these devices are calibrated taking as reference the universal law of the wall, they cannot be reliably used in regions where the existence of the law of the wall can be questioned. Wygnanski et al. estimated the skin-friction through three different techniques: a momentum integral method, the mean velocity gradient in the viscous sub-layer, and by use of a Preston tube.

The establishment of the above concepts for the velocity field clearly raises some questions for the temperature field. An immediate question concerns the existence of an appropriate temperature scale at the outer edge of the equilibrium layer. At the point of velocity maximum (U_M), the temperature profiles reach a minimum (T_m) (Guerra et al., 2005). Thus, drawing an analogy to the velocity analyses of Narasimha et al. (1973) and of Özdemir and Whitelaw (1992), one would expect the appropriate scaling temperature parameter to be this minimum temperature.

The law of the wall for the temperature profile can then be written as

$$t^+ = \varkappa_t^{-1} \ln y^+ + B((T_w - T_m)/t^*) \tag{73}$$

where t* is the friction temperature, T_W the wall temperature and $\varkappa_t$ is the von Karman constant for the temperature field.

In Guerra et al. (2005) a detailed analysis is made about the parametric behavior of A and B for an impinging jet.

These results were recently revisited by Loureiro and Silva Freire (2012). The analysis in both works resort to structural arguments obtained through the intermediate variable technique.

11 Conclusions

The intermediate variable technique of Kaplun(1967) has been formally applied to perturbed turbulent boundary layers to develop near-wall local solutions. The technique is also useful to identify matching difficulties that eventually appear in multi-layered problems. The compressible turbulent boundary layer is a typical example. For some time, it was believed that the matched asymptotic expansions method would not result in a complete matching for the velocity, temperature and density profiles; it was thought that the number of available parameters was not sufficient to satisfy all the required matching conditions. However, using the intermediate variable technique Silva Freire (1989) showed that formally this is not true. In particular, it was shown that the matching difficulties resulted from a faulty choice of the asymptotic expansions.

Even extremely difficult problems, including that of the interaction of a shock wave with a turbulent boundary layer can benefit from application of the method of Kaplun (see Silva Freire (1988b)). The intermediate variable technique can be easily applied to show how the vicinity of the shock defines a purely inviscid region; the viscous effects are shown to be confined just to the viscous sublayer. The developments can be used to derive a skin-friction equation that embodies effects of the shock strength, the state of equilibrium of the boundary layer and transpiration at the wall.

Another problem of evident interest is the derivation of near wall solutions for two-equation differential models ($\kappa - \epsilon$, $\kappa - \omega$,$\kappa - \Omega$). In Avelino et al. (1999), local solutions for the fully turbulent region of flows subject to wall transpiration are developed for the description of $\hat{e}$ and ϵ. The expressions were implemented in numerical codes, furnishing very good results against some reference experimental data.

Conditions where the boundary layer is in a non-equilibrium state resulting from changes in surface properties must also be studied in connection with modifications in the viscous sublayer. As a flow moves, say, from a rough to a smooth surface the properties of the outer region remain nearly unaltered. The near wall flow, however, adjusts immediately to the new wall condition. There remains an intermediate layer - the fully turbulent region - that needs to match these two apparently conflicting conditions through a slow variation. In Avelino and Silva Freire (2002) and Loureiro et al. (2010) this problem has been address in light of the developments of the previous sections.

The results described for flows over rough walls have wide application. The prediction of friction factors and Nusselt numbers for turbulent forced convection in rod bundles with smooth and rough surfaces is a recognized difficult problem. The present approach permits (Su and Silva Freire, 2002) the development of analytical methods that are easy to implement and very accurate.

The mechanics of turbulent drag reduction for flow over riblet surfaces has been much discussed for attached flows. The effect of riblets on flow separation over steep, smooth and rough curved surfaces is a completely open issue. In Loureiro and Silva Freire (2011c), four types of two-dimensional surfaces were studied to understand the modifications in the near structure. Results were analyzed in view of the asymptotic structure described in terms of the intermediate variable technique.

The purpose of the present compilation has been to show the usefulness of the intermediate variable technique. In particular, the method has been applied to a host of problems to illustrate how structural as well as quantitative information can be gained.

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