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CONJUGATE HEAT TRANSFER: ANALYSIS VIA INTEGRAL TRANSFORMS AND EIGENVALUE PROBLEMS

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An integral transform approach to the solution of the problem on conjugate heat transfer, combining the single-domain formulation with the convective eigenfunction expansion basis within the total integral transformation framework, which leads to a nonclassical eigenvalue problem, is presented. The problem on the conjugate heat transfer in the transient two-dimensional incompressible laminar flow of a Newtonian fluid in a parallel-plate channel is considered to illustrate the hybrid numerical-analytical approach. To demonstrate the improvement of the convergence rate achieved with the methodology proposed, a critical comparison against the traditional total integral transformation solution of the diffusive eigenvalue problem is provided, and results are presented and discussed for three representative situations realized with different Peclet numbers: Pe = 1, 10 and 100. A remarkable improvement of the convergence rate, obtained especially with the large Péclet numbers, offers evidence of the validity of the expansion constructed upon the nonclassical eigenvalue problem proposed.

Keywords: *conjugate heat transfer, internal convection, single-domain formulation, convective eigenfunction expansion basis, integral transforms, nonself-adjoint eigenvalue problem.*

Introduction. Simulation and analytical description of the conjugate conduction–convection heat transfer in the fluid flow in a channel, defined by the partial differential equations for the energy balances in the fluid and in the solid channel walls, is a fundamental mathematical problem of thermal sciences. The goal was also to solve the coupled differential equations and avoid the errors inherent in approximate formulations that either disregard the participation of the solid walls in the thermal process or simplify the participation of the fluid in it by accounting for the convective heat transfer through the adoption of correlated results for the coefficients of heat transfer under limiting conditions. The A. V. Luikov Heat and Mass Transfer Institute (Minsk, Belarus) pioneered and guided research efforts in this area, as illustrated by the early works of Perelman [1] and Luikov et al. [2], which presented analytical solutions for simplified models in the situations with both internal and external heat flows.

In recent years, problems on conjugate heat transfer have been revisited by Knupp et al. [3–5], which was mainly motivated by the microscale heat transfer applications [5–9]. A computational-analytical approach to the solution of conjugate problems through the combination of integral transforms with the single-domain strategy was then advanced. Basically this strategy involves the reformulation of the problem on the conjugate heat transfer in the fluid flow in a channel to be solved by rewriting the energy balances for the solid and fluid regions as a single partial differential equation for the whole spatial domain, whose parameters are represented as space variable functions with abrupt transitions at the fluid–wall interface so as to take into account the transition of the solid and fluid physical properties. Then the generalized integral transform technique (GITT) [10] was recalled to provide a hybrid numerical-analytical solution of this single energy equation for the whole domain. In most contributions using this solution so far, the energy equation is integrally transformed only in the direction transversal to the flow, and the resulting transformed partial differential system is then solved numerically. This partial transformation scheme is preferred to avoid the slower convergence of the eigenfunction expansions in the longitudinal direction, since the convective term would be actually treated as a source term if the traditional total transformation scheme is employd. To overcome the slower convergence in the total transformation of highly convective functions, Cotta et al. [11] recently advanced a computational-analytical approach to the solution of convection–diffusion problems via integral

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transforms, incorporating convective effects into the chosen eigenvalue problem that forms the basis of the eigenfunction expansion proposed, bringing a new perspective for the total transformation of conjugate problems under the single-domain formulation.

The present work is directed towards the indicated direction and presents a combination of the single domain formulation with the convective eigenvalue problem within the total integral transformation framework. As an illustration, the problem on the conjugate heat transfer in the transient two-dimensional fluid flow in a parallel-plate channel is considered [4]. For the purpose of demonstrating the improvement in the convergence rate achieved with the new methodology, the traditional total integral transformation solution method, employing the classical diffusive eigenvalue problem, is also presented. In both solution alternatives there arise eigenvalue problems with arbitrary space variable coefficients, which are solved through the GITT.

Formulation of the Problem. The problem investigated here is similar to the problem considered in [4]. The problem involves an incompressible internal laminar flow of a Newtonian fluid between parallel plates, which undergoes a convective heat transfer due to the temperature T_w prescribed at the outer surface of the channel wall. The channel wall participates in the heat transfer through the transverse and longitudinal heat conductions. The fluid flow at the input of the channel has a temperature T_{in} and a completely developed velocity profile $u_f(y)$ (Fig. 1). The flow is assumed to be dynamically developed and thermally developing.

The conjugate heat transfer in the two-dimensional fluid flow in the parallel-plate channel is defined by a single-domain model with coefficients representing variable space functions experiencing abrupt transitions at the fluid–solid wall interface [3–5]. The problem on the conjugate heat transfer in the transient fluid flow, symmetric at $y = 0$, in the single-domain formulation with space variable coefficients has the form

$$w(y)\,\frac{\partial T(x,\,y,\,t)}{\partial t} + u(y)w_f\,\frac{\partial T(x,\,y,\,t)}{\partial x} = k(y)\,\frac{\partial^2 T}{\partial x^2} + \frac{\partial}{\partial y}\left(k(y)\,\frac{\partial T}{\partial y}\right), \tag{1}$$

$$0 < y < y_w\,, \quad 0 < x < L\,, \quad t > 0\,,$$

$$T(0,\,y,\,t) = T_{in}\,, \quad \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0\,, \tag{2}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0\,, \quad T(x,\,y_w,\,t) = T_w\,, \tag{3}$$

$$T(x,\,y,\,0) = T_0\,, \tag{4}$$

$$u(y) = \begin{cases} u_f(y)\,, & 0 < y < y_f/2 \\ 0\,, & y_f/2 < y < y_w \end{cases}, \tag{5}$$

$$k(y) = \begin{cases} k_f\,, & 0 < y < y_f/2 \\ k_s\,, & y_f/2 < y < y_w \end{cases}, \tag{6}$$

$$w(y) = \begin{cases} w_f\,, & 0 < y < y_f/2 \\ w_s\,, & y_f/2 < y < y_w \end{cases}. \tag{7}$$

Introducing the dimensionless variables

$$X = \frac{x/y_w}{\mathrm{Re\ Pr}} = \frac{x}{y_w \mathrm{Pe}}\,, \quad Y = \frac{y}{y_w}\,, \quad U = \frac{u}{4u_{av}}\,, \quad \theta = \frac{T - T_{in}}{T_0 - T_{in}}\,, \quad K = \frac{k}{k_f}\,, \quad W = \frac{w}{w_f}\,,$$
$$\mathrm{Re} = \frac{u_{av} 4 y_w}{\nu}\,, \quad \mathrm{Pr} = \frac{\nu}{\alpha}\,, \quad \mathrm{Pe} = \mathrm{Re\ Pr} = \frac{u_{av} 4 y_w}{\alpha}\,, \quad \alpha = \frac{k_f}{w_f}\,, \quad \tau = \frac{\alpha t}{y_w^2}\,, \tag{8}$$

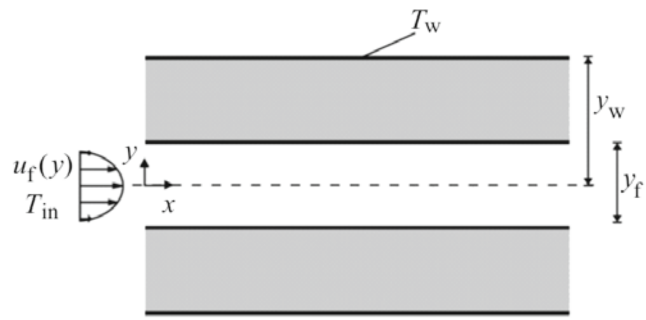

Fig. 1. Schematic representation of the conjugated problem.

we represent the problem in the dimensionless formulation for $T_w = T_{in}$:

$$W(Y)\,\frac{\partial\theta}{\partial\tau} + U(Y)\,\frac{\partial\theta}{\partial X} = \frac{K(Y)}{\mathrm{Pe}^2}\,\frac{\partial^2\theta}{\partial X^2} + \frac{\partial}{\partial Y}\left(K(Y)\,\frac{\partial\theta}{\partial Y}\right), \quad 0 < Y < 1\,, \quad 0 < X < L^*\,, \quad \tau > 0\,, \tag{9}$$

$$\theta(0, Y, \tau) = 0\,, \quad \left.\frac{\partial\theta}{\partial X}\right|_{X=L^*} = 0\,, \tag{10}$$

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = 0\,, \quad \theta(X, 1, \tau) = 0\,, \tag{11}$$

$$\theta(X, Y, 0) = 1\,, \tag{12}$$

$$U(Y) = \begin{cases} U_f(Y)\,, & 0 < Y < Y_i = y_f/(2y_w) \\ 0\,, & Y_i < Y < 1 \end{cases}\,, \tag{13}$$

$$K(Y) = \begin{cases} 1\,, & 0 < Y < Y_i = y_f/(2y_w) \\ k_s/k_f\,, & Y_i < Y < 1 \end{cases}\,, \tag{14}$$

$$W(Y) = \begin{cases} 1\,, & 0 < Y < Y_i = y_f/(2y_w) \\ w_s/w_f\,, & Y_i < Y < 1 \end{cases}\,. \tag{15}$$

Solution of the Problem. Two ways of solution of problem (9)–(15) via integral transforms were investigated. The first way is realized with the use of the traditional GITT [10]. It involves consideration of the classical diffusive auxiliary eigenvalue problem with space variable coefficients (representing a special case of the general Sturm–Liouville problem), thus transferring the information on the space variation of the original problem coefficients to the eigenfunction expansion basis but not taking into account any information regarding the convective term treated as a source term. The second methodology is based on the reformulation proposed by Cotta et al. [11] in which the auxiliary eigenvalue problem incorporates convective effects in addition to the information on the space variation of the coefficients of the original problem in the single-domain formulation, which, in the present case, leads to the nonclassical eigenvalue problem. Both eigenvalue problems do not allow for explicit analytical solution, and the GITT itself is employed [5, 10].

Solution via the diffusive eigenvalue problem. Following the traditional GITT formalism for the total transformation scheme, the transform–inverse pair used for solving problem (9)–(15) is defined as

$$\bar{\theta}_i(\tau) = \int_0^1\int_0^{L^*} W(Y)\tilde{\psi}_i(X, Y)\theta(X, Y, \tau)\,dX\,dY\,, \tag{16}$$

$$\theta(X, Y, \tau) = \sum_{i=1}^{\infty} \overline{\theta}_i(\tau)\tilde{\psi}_i(X, Y) , \quad (17)$$

with the normalized eigenfunction and norms

$$\tilde{\psi}_i(X, Y) = \frac{\psi_i(X, Y)}{\sqrt{N_i}} , \quad (18)$$

$$N_i = \int_0^1 \int_0^{L^*} W(Y)\psi_i^2(X, Y)\, dX\, dY . \quad (19)$$

The following eigenvalue problem has been obtained by direct application of the separation of variables to the purely diffusive version of problem (9)–(15) so that the information concerning the transition of the two original domains $W(Y)$ and $K(Y)$ is accounted for by eigenvalues and eigenfunctions:

$$\frac{K(Y)}{\mathrm{Pe}^2} \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial}{\partial Y}\left(K(Y) \frac{\partial \psi}{\partial Y}\right) + \mu^2 W(Y)\psi , \quad 0 < Y < 1 , \quad 0 < X < L^* , \quad (20)$$

$$\psi(0, Y) = 0 , \quad \left.\frac{\partial \psi}{\partial X}\right|_{X=L^*} = 0 , \quad (21)$$

$$\left.\frac{\partial \psi}{\partial Y}\right|_{Y=0} = 0 , \quad \psi(X, 1) = 0 . \quad (22)$$

Problem (20)–(22) is the classical Sturm–Liouville problem which does not allow for explicit analytic solution because of the presence of the space variable coefficients $W(Y)$ and $K(Y)$ in it. Nonetheless, the GITT itself can be employed to provide a hybrid numerical-analytical solution for this eigenvalue problem [5, 10]. Once this solution is made available, problem (9)–(15) is integrally transformed with the operator $\int_0^1 \int_0^{L^*} \tilde{\psi}_i(X, Y)(\cdot)\, dX\, dY$, yielding the problem

$$\frac{d\overline{\theta}_i(\tau)}{d\tau} + \mu_i^2 \overline{\theta}_i(\tau) = \overline{g}_i(\tau, \overline{\boldsymbol{\theta}}) , \quad i = 1, 2, \dots , \quad (23)$$

$$\overline{g}_i(\tau, \overline{\boldsymbol{\theta}}) = -\sum_{j=1}^{\infty} \overline{\theta}_j(\tau) \int_0^1 \int_0^{L^*} U(Y)\tilde{\psi}_i(X, Y) \frac{\partial \tilde{\psi}_j(X, Y)}{\partial X}\, dX\, dY \quad (24)$$

with the transformed initial conditions

$$\overline{\theta}_i(0) = \int_0^1 \int_0^{L^*} W(Y)\tilde{\psi}_i(X, Y)\, dX\, dY . \quad (25)$$

Equations (23)–(25) form an infinite system of coupled ordinary differential equations, which can be solved analytically by an appropriate method of algebraic eigensystem analysis after the truncation to an order N or numerically using solvers of the initial value problem with error control to determine the transformed temperatures $\overline{\theta}_i(\tau)$. The Mathematica platform [12] provides the routine NDSolve for numerical solution with automatic control of absolute and relative errors. Once the transformed potentials are numerically computed, the Mathematica routine automatically provides an interpolating function object that approximates the behavior of a solution depending on the variable τ in the continuous form. Then, the inversion formula (17) can be recalled to yield the dimensionless temperature θ(X, Y, τ) at any desired position (X, Y) and time τ.

Solution via the convective eigenvalue problem. Following the formalism introduced by Cotta et al. [11], we transform the coefficients in problem (9)–(15) with the use of the exponential factors

$$K_X(X, Y) = \mathrm{e}^{-\int \mathrm{Pe}^2 U_X^* \, dX} \,, \quad K_Y(Y) = \mathrm{e}^{-\int U_Y^* \, dY} \,, \tag{26}$$

$$U_X^* = \frac{U(Y)}{K(Y)} \,, \quad U_Y^* = -\frac{1}{K(Y)} \frac{dK(Y)}{dY} \tag{27}$$

and, as a result, obtain problem (9)–(15) in the generalized diffusive form

$$W \frac{\partial \theta}{\partial \tau} = \frac{K_Y}{\mathrm{Pe}^2} \frac{\partial}{\partial X} \left(K_X \frac{\partial \theta}{\partial X} \right) + K_X \frac{\partial}{\partial Y} \left(K_Y \frac{\partial \theta}{\partial Y} \right), \quad 0 < Y < 1 \,, \quad 0 < X < L^* \,, \quad \tau > 0 \,, \tag{28}$$

$$\theta(0, Y, \tau) = 0 \,, \quad \left. \frac{\partial \theta}{\partial X} \right|_{X = L^*} = 0 \,, \tag{29}$$

$$\left. \frac{\partial \theta}{\partial Y} \right|_{Y = 0} = 0 \,, \quad \theta(X, 1, \tau) = 0 \,, \tag{30}$$

$$\theta(X, Y, 0) = 1 \,, \tag{31}$$

$$W(X, Y) = \frac{W(Y) K_x(X, Y) K_Y(Y)}{K(Y)} \,. \tag{32}$$

Separation of variables in problem (28)–(32) with $\theta(X, Y) = \Gamma(t)\xi(X, Y)$ yields the problem

$$\frac{d\Gamma(\tau)}{d\tau} + \beta^2 \Gamma(\tau) = 0 \,, \quad \Gamma(t) = A \mathrm{e}^{-\beta^2 \tau} \tag{33}$$

and the nonclassical eigenvalue problem

$$\frac{K_Y}{\mathrm{Pe}^2} \frac{\partial}{\partial X} \left(K_X \frac{\partial \zeta}{\partial X} \right) + K_X \frac{\partial}{\partial Y} \left(K_Y \frac{\partial \zeta}{\partial Y} \right) + \beta^2 W \zeta = 0 \,, \quad 0 < Y < 1 \,, \quad 0 < X < L^* \,, \tag{34}$$

$$\zeta(0, Y) = 0 \,, \quad \left. \frac{\partial \zeta}{\partial X} \right|_{X = L^*} = 0 \,, \tag{35}$$

$$\left. \frac{\partial \zeta}{\partial Y} \right|_{Y = 0} = 0 \,, \quad \zeta(X, 1) = 0 \,. \tag{36}$$

The eigenvalue problem given by Eqs. (34)–(36) is not self-adjoint, and the eigenfunctions $\zeta_i(X, Y)$, $i = 1, 2, 3, \ldots$, do not follow one and the same orthogonality property as for the classical Sturm–Liouville problem. Moreover, the corresponding eigenvalue spectrum is not known a priori, and complex quantities can be present eventually in it. There are a number of works dealing with analytical solution employing different nonclassical eigenvalue problems on heat transfer [12–21], but unfortunately none correspond to the present formulation, and the majority of them do not attempt to prove the assumption that the eigenfunctions obtained originate a complete set, which requires finding an associated adjoint problem, and essentially verify a solution by comparison with alternative numerical solutions. Therefore, just assuming that the eigenvalue problem given by Eqs. (34)–(36) originates a complete set of eigenfunctions, we write a solution of problem (28)–(32) in the form

$$\theta(X, Y, \tau) = \sum_{i=1}^{\infty} A_i \zeta_i(X, Y) e^{-\beta_i^2 t} , \tag{37}$$

where the expansion coefficients A_i are determined from the initial condition. Hence, operating on Eq. (10) with $\int_0^1 \int_0^{L^*} \zeta_j(X, Y)(\cdot)\, dX\, dY$ at $\tau = 0$, we arrive at the system of linear algebraic equations

$$\int_0^1 \int_0^{L^*} \zeta_j(X, Y)\, dX\, dY = \sum_{i=1}^{\infty} A_i \int_0^1 \int_0^{L^*} \zeta_i(X, Y) \zeta_j(X, Y)\, dX\, dY , \quad i = 1, 2, 3 \ldots , \quad j = 1, 2, 3 \ldots \tag{38}$$

which, when truncated to a finite order N, can be solved for the coefficients A_i, and the expansion given by Eq. (37) can be used to calculate the dimensionless temperature θ at any position (X, Y) and time τ.

Solution of eigenvalue problems via integral transforms. The eigenvalue problems given by Eqs. (20)–(22) and Eqs. (34)–(36) do not allow for explicit analytical solution. Hence the GITT [10] is employd. For both problems, we use an auxiliary basis given by the simpler self-adjoint eigenvalue problem

$$\frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2} + \lambda^2 \Omega , \quad 0 < Y < 1 , \quad 0 < X < L^* , \tag{39}$$

$$\Omega(0, Y) = 0 , \quad \left. \frac{\partial \Omega}{\partial X} \right|_{X = L^*} = 0 , \tag{40}$$

$$\left. \frac{\partial \Omega}{\partial Y} \right|_{Y=0} = 0 , \quad \Omega(X, 1) = 0 \tag{41}$$

and solve problems (20)–(22) and (34)–(36) , respectively, with the integral inverse–transform pairs

$$\psi_i(X, Y) = \sum_{m=0}^{\infty} \tilde{\Omega}_n(X, Y) \bar{\psi}_{i,m} , \tag{42}$$

$$\bar{\Psi}_{i,m} = \int_0^1 \int_0^{L^*} \psi_i(X, Y) \tilde{\Omega}_m(X, Y)\, dX\, dY , \tag{43}$$

$$\zeta_i(X, Y) = \sum_{m=0}^{\infty} \tilde{\Omega}_n(X, Y) \bar{\zeta}_{i,m} , \tag{44}$$

$$\bar{\zeta}_{i,m} = \int_0^1 \int_0^{L^*} \zeta_i(X, Y) \tilde{\Omega}_m(X, Y)\, dX\, dY . \tag{45}$$

Then, operating on Eqs. (20) and (34) with $\int_0^1 \int_0^{L^*} \Omega_j(X, Y)(\cdot)\, dX\, dY$ and making use of the inversion formulae (42) and (44), we obtain, respectively, the algebraic problems

$$(\mathbf{A} - \mu^2 \mathbf{B}) \bar{\boldsymbol{\psi}} = 0 , \tag{46}$$

$$A_{ij} = \int_0^1 \int_0^{L^*} \frac{K}{\mathrm{Pe}^2} \frac{\partial^2 \tilde{\Omega}_i}{\partial X^2} \tilde{\Omega}_j \, dX\,dY + \int_0^1 \int_0^{L^*} \frac{\partial}{\partial Y} \left(K \frac{\partial \tilde{\Omega}_i}{\partial Y} \right) \tilde{\Omega}_j \, dX\,dY \,, \tag{47}$$

$$B_{ij} = -\int_0^1 \int_0^{L^*} W \tilde{\Omega}_i \tilde{\Omega}_j \, dX\,dY \,, \tag{48}$$

$$(\mathbf{C} - \beta^2 \mathbf{D}) \bar{\zeta} = 0 \,, \tag{49}$$

$$C_{ij} = \int_0^1 \int_0^{L^*} \frac{K_Y}{\mathrm{Pe}^2} \frac{\partial}{\partial X} \left(K_X \frac{\partial \tilde{\Omega}_i}{\partial X} \right) \tilde{\Omega}_j \, dX\,dY + \int_0^1 \int_0^{L^*} K_X \frac{\partial}{\partial Y} \left(K_Y \frac{\partial \tilde{\Omega}_i}{\partial Y} \right) \tilde{\Omega}_j \, dX\,dY \,, \tag{50}$$

$$D_{ij} = -\int_0^1 \int_0^{L^*} W \tilde{\Omega}_i \tilde{\Omega}_j \, dX\,dY \,. \tag{51}$$

The algebraic problems (46) and (49) can be numerically solved on truncation to a sufficiently large finite order M for the eigenvalues μ^2 and β^2 and the eigenvectors $\bar{\boldsymbol{\psi}}$ and $\bar{\boldsymbol{\zeta}}$, respectively. The eigenvectors $\bar{\boldsymbol{\psi}}$ form the expansion coefficients for the eigenfunctions $\psi(X, Y)$ in Eq. (42), and the eigenvectors $\bar{\boldsymbol{\zeta}}$, form the expansion coefficients for the eigenfunctions $\zeta(X, Y)$ in Eq. (44). It should be noted that the matrix **C** related to the nonclassical eigenvalue problem (34)–(36) is not symmetric unlike the matrix **A** related to the classical eigenvalue problem (20)–(22), which is symmetric as expected, always yielding real eigenvalues.

Results and Discussion. As a test case for obtaining numerical results, we considered the problem on the conjugate heat transfer in the water flow in a parallel-plate channel with acrylic walls [4] at $k_s/k_f = 0.25$ and $w_s/w_f = 0.35$. The fluid–wall interface was set at the dimensionless position $Y_i = 0.5$, and the coefficients $U(Y)$, $W(Y)$, and $K(Y)$ with abrupt transitions were simulated using the expressions

$$\phi(Y) = \phi_f + (\phi_s - \phi_f)\delta(Y) \,, \quad \delta(Y) = \frac{1}{1 + e^{-\gamma(Y - Y_i)}} \,, \tag{52}$$

where ϕ represents the desired coefficients $U(Y)$, $W(Y)$ or $K(Y)$. Equations (52) provide an abrupt, yet smooth, transition at the position Y_i where the parameter γ controlling the abruptness of this transition has the value $\gamma = 500$ adopted by the numerical results presented here. The use of a smooth function for simulation of abrupt transitions of the indicated coefficients presents some advantages because derivatives of these coefficients appear in Eqs. (27), (47), and (50), which was demonstrated in the previous works [22, 23].

Numerical results are now presented to provide comparisons regarding the convergence of the eigenfuncion expansions of the integral-transform solutions via the diffusive and convective eigenvalue problems. In all the cases, $M = 75$ was used as the truncation order in the eigenvalue problem solution via the GITT. For further verification, independent purely numerical results obtained with the commercial CFD solver Comsol Multiphysics are also presented. In this solver, the conjugate problem is treated classically (unlike the single-domain formulation) with the coupling of the two physical effects (the heat transfer in the solid and the heat transfer in the fluid) assigned previously to the channel wall and the fluid flow in it, and a mesh is generated automatically under the "extremely refined" option of the physically controlled mesh-generation features.

Three representative situations were analyzed: Pe = 1, 10, and 100, starting with the smallest Péclet number, at which the convective term is less significant and the traditional solution of the diffusive eigenvalue problem via integral transforms is expected to show a good convergence behavior. This case can also be used to present a co-verification with the novel solution way via the convective eigenvalue problem. Table 1 illustrates the convergence behavior of the solution employing the convective eigenvalue problem for Pe = 1, demonstrating that only with $N = 18$ a complete convergence of five to six

TABLE 1. Convergence Behavior of the Calculated Temperatures at $X\text{Pe} = 0.05$, $\tau = 0.01$, and $\text{Pe} = 1$

N	$Y = 0.2$	$Y = 0.3$	$Y = 0.4$	$Y = 0.6$	$Y = 0.7$	$Y = 0.8$
Convective eigenvalue problem						
3	0.256079	0.252575	0.256287	0.328441	0.330669	0.269604
6	0.257835	0.259193	0.263403	0.304034	0.307831	0.284545
9	0.257596	0.259685	0.264093	0.310626	0.316237	0.290713
12	0.259063	0.260886	0.265352	0.310019	0.315476	0.290549
15	0.259085	0.260927	0.265346	0.309965	0.315517	0.290559
18	0.259077	0.260928	0.265353	0.309960	0.315529	0.290546
21	0.259077	0.260928	0.265353	0.309961	0.315527	0.290547
24	0.259077	0.260928	0.265353	0.309964	0.315531	0.290551
Diffusive eigenvalue problem						
3	0.258367	0.254450	0.257331	0.328549	0.330834	0.269522
6	0.260024	0.260914	0.264427	0.304115	0.307841	0.284516
9	0.259803	0.261335	0.265009	0.310101	0.315886	0.290971
12	0.259409	0.261247	0.265544	0.310081	0.315537	0.290504
15	0.259315	0.261151	0.265533	0.310025	0.315506	0.290556
18	0.259356	0.261113	0.265515	0.309998	0.315517	0.290556
21	0.259349	0.261126	0.265501	0.309993	0.315520	0.290555
24	0.259355	0.261129	0.265489	0.309970	0.315530	0.290551
27	0.259353	0.261131	0.265485	0.309946	0.315552	0.290555
30	0.259139	0.260969	0.265377	0.310018	0.315505	0.290571
33	0.259115	0.260972	0.265388	0.309979	0.315533	0.290556
36	0.259120	0.260970	0.265385	0.309975	0.315535	0.290559
39	0.259126	0.260966	0.265380	0.309970	0.315533	0.290554
42	0.259127	0.260968	0.265376	0.309967	0.315535	0.290553
45	0.259127	0.260967	0.265377	0.309965	0.315534	0.290554
48	0.259128	0.260966	0.265374	0.309963	0.315534	0.290553
51	0.259127	0.260966	0.265374	0.309964	0.315535	0.290551
54	0.259127	0.260966	0.265372	0.309963	0.315533	0.290552
57	0.259090	0.260932	0.265344	0.309989	0.315510	0.290574
60	0.259079	0.260930	0.265356	0.309968	0.315533	0.290551
63	0.259080	0.260929	0.265356	0.309968	0.315533	0.290552
66	0.259080	0.260930	0.265356	0.309968	0.315533	0.290552
69	0.259080	0.260930	0.265357	0.309969	0.315533	0.290552

significant digits is attained at the selected positions. The convergence behavior regarding the traditional solution method via the diffusive eigenvalue problem is also illustrated in this table, and it is substantially slower even in this small Péclet number scenario, demanding $N = 60$ to achieve a similar convergence pattern. As the complete convergence was achieved in both solution ways, it is interesting to observe that these solutions correlate with each other to five to six significant digits. Figure 2a and b

TABLE 2. Convergence Behavior of the Calculated Temperatures at XPe = 0.05, τ = 0.01, and Pe = 10

N	$Y = 0.2$	$Y = 0.3$	$Y = 0.4$	$Y = 0.6$	$Y = 0.7$	$Y = 0.8$
Convective eigenvalue problem						
3	0.210106	0.217457	0.234580	0.323458	0.328714	0.272011
6	0.211044	0.223159	0.241558	0.302504	0.307638	0.284495
9	0.210796	0.223583	0.242205	0.309022	0.316087	0.290680
12	0.212114	0.224704	0.243447	0.308494	0.315314	0.290522
15	0.212132	0.224737	0.243437	0.308459	0.315344	0.290520
18	0.212124	0.224738	0.243443	0.308454	0.315355	0.290508
21	0.212125	0.224737	0.243444	0.308457	0.315358	0.290512
24	0.212125	0.224737	0.243444	0.308458	0.315358	0.290512
Diffusive eigenvalue problem						
3	0.229845	0.233831	0.243907	0.326646	0.331598	0.269901
6	0.230173	0.238731	0.251185	0.303308	0.307737	0.284213
9	0.230108	0.238477	0.250818	0.304177	0.312796	0.293083
12	0.214841	0.227951	0.245312	0.308835	0.316078	0.290226
15	0.213696	0.226549	0.245162	0.309034	0.315266	0.290469
18	0.214189	0.226154	0.244914	0.308830	0.315243	0.290606
21	0.214114	0.226286	0.244763	0.308769	0.315288	0.290585
24	0.214174	0.226318	0.244649	0.308533	0.315355	0.290516
27	0.214154	0.226339	0.244605	0.308313	0.315556	0.290543
30	0.212690	0.225221	0.243833	0.308800	0.315297	0.290557
33	0.212465	0.225154	0.243788	0.308600	0.315382	0.290564
36	0.212516	0.225135	0.243753	0.308565	0.315395	0.290592
39	0.212581	0.225096	0.243701	0.308523	0.315380	0.290547
42	0.212590	0.225115	0.243665	0.308490	0.315402	0.290536
45	0.212589	0.225110	0.243670	0.308470	0.315389	0.290541
48	0.212598	0.225100	0.243648	0.308452	0.315392	0.290531
51	0.212583	0.225100	0.243643	0.308457	0.315402	0.290516
54	0.212587	0.225099	0.243631	0.308452	0.315385	0.290528
57	0.212280	0.224821	0.243404	0.308662	0.315192	0.290705
60	0.212187	0.224795	0.243488	0.308501	0.315384	0.290519
63	0.212198	0.224785	0.243488	0.308501	0.315386	0.290528
66	0.212207	0.224788	0.243486	0.308498	0.315381	0.290527
69	0.212205	0.224790	0.243497	0.308503	0.315383	0.290527

illustrates the convergence behavior of solutions by presenting their transversal temperature profiles, calculated with different truncation orders (N = 3, 6, 9 and 12). It is seen that N = 9 is enough to deliver curves fully converged to the graph scale in both solution ways. Figure 2c demonstrates the excellent agreement between the solutions obtained by the integral transform

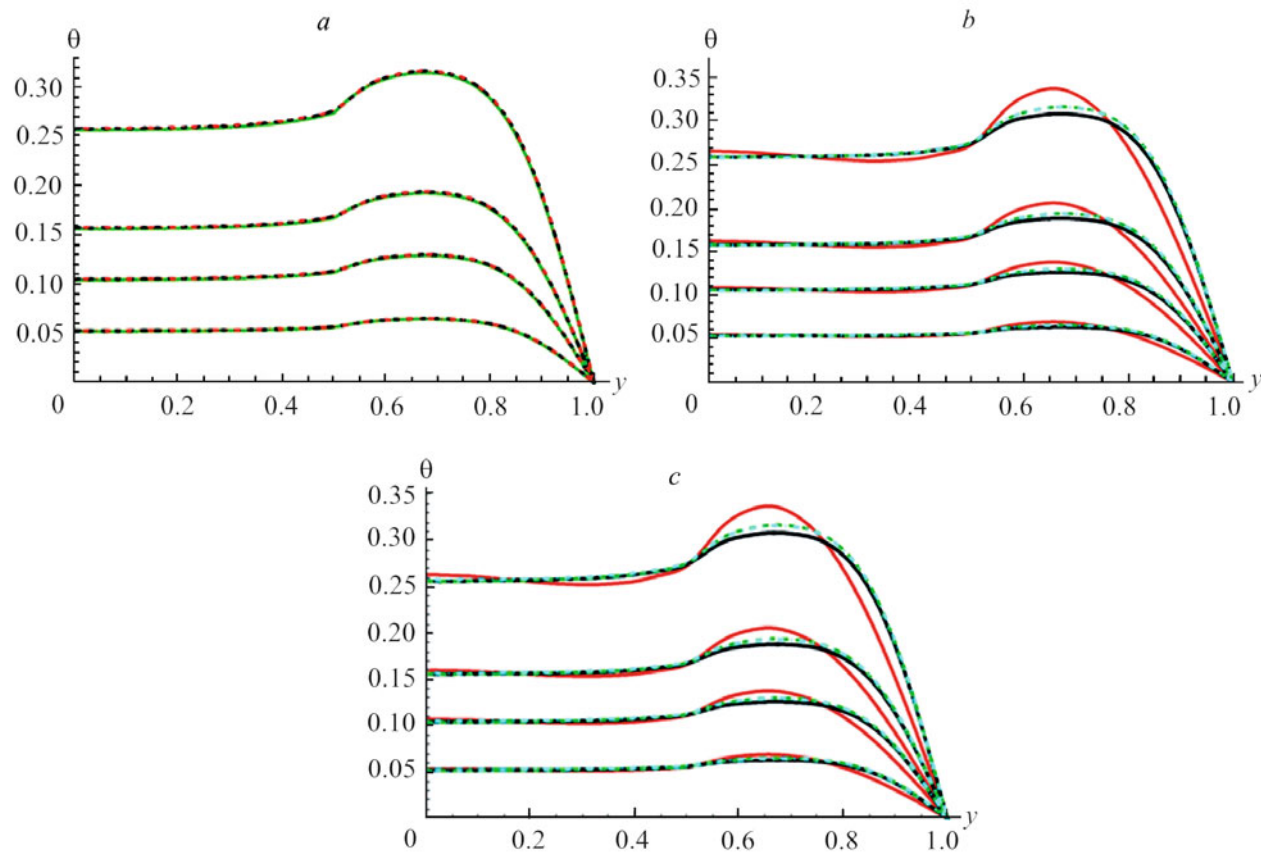

Fig. 2. Transversal temperature profiles calculated employing the convective eigenvalue problem (a) and the diffusive eigenvalue problem (b) with $N = 3$ (red lines), 6 (black lines), 9 (green dashed lines) and 12 (cyan dash-dotted lines) and employing the Comsol Multiphysics (green lines), the convective eigenvalue problem with $N = 9$ (black dashed lines), and the diffusive eigenvalue problem with $N = 69$ (red dash-dotted lines) at Pe =1.

method with the purely numerical solution obtained using the Comsol Multiphysics solver, providing verification of the single-domain formulations adopted here.

As Pe = 10, the convergence behavior of the solution via the convective eigenvalue problem is practically identical to that obtained in the previous case, also achieving a convergence to five to six significant digits at the selected positions, as demonstrated in Table 2, reconfirming the importance of the incorporation of the convective term into the eigenvalue problem. On the other hand, the convergence behavior of the classical solution, employing the diffusive eigenvalue problem, slows down substantially, achieving approximately four significant digits at $N = 69$. This result increases the importance of the convective term treated as a source term in the traditional solution. The graphical illustration of the convergence behavior, presented in Fig. 3a and b, is also very interesting. While the solution employing the convective eigenvalue problem shows a complete convergence to the graph scale only $N = 9$, the traditional solution fails to achieve good results with such a small truncation order. It should be also noted that the convergence behavior of the classical solution is much worse in the fluid region ($0 < Y < 0.5$), especially near $Y = 0$, where the fluid flow has higher velocities because of the increase in the Péclet number. In Fig. 3c the solutions via the convective eigenvalue problem with $N = 9$ and the solution via the diffusive eigenvalue problem with $N = 69$ are also compared to that obtained using the Comsol Multiphysics solver, showing an excellent agreement.

The case where Pe = 100 was analyzed by the data presented in Table 3 and Fig. 4 by analogy with the previous cases, but the difference between the solutions via the convective and diffusive eigenvalue problems in convergence behavior

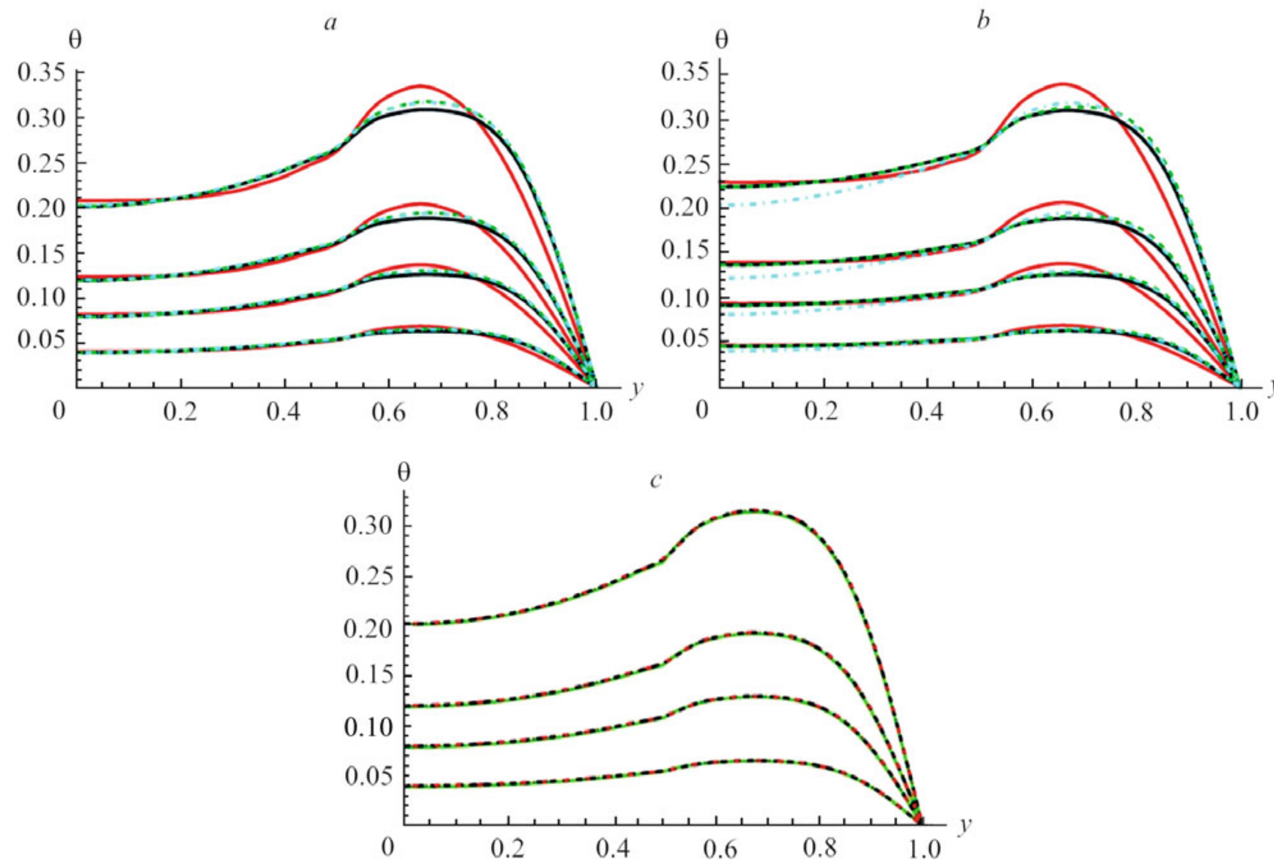

Fig. 3. Transversal temperature profiles calculated employing the convective eigenvalue problem (a) and the diffusive eigenvalue problem (b) with $N = 3$ (red lines), 6 (black lines), 9 (green dashed lines) and 12 (cyan dash-dotted lines) and employing the Comsol Multiphysics (green lines), the convective eigenvalue problem with $N = 9$ (black dashed lines), and the diffusive eigenvalue problem with $N = 69$ (red dash-dotted lines) at Pe =10.

was even larger. While the solution via the convective eigenvalue problem converges completely to five to six significant digits at a truncation order as small as $N = 18$, the classical solution via the diffusive eigenvalue problem converges only to two to three significant digits at the largest truncation order $N = 69$ considered in this work. The slower convergence of the traditional solution is clearly seen from Fig. 4b: the calculated temperature profiles do not converge to the graph scale at the truncation order $N = 9$ required for the solution via the convective eigenvalue problem to converge to the graph scale (Fig. 4a). Nonetheless, even in this highly convective situation, if larger truncation orders are used ($N = 69$), the classical solution also succeeds in graphical matching the solution via the convective eigenvalue problem and the solution obtained using Comsol Multiphysics solver (Fig. 4c).

Conclusions. We combined the single-domain formulation strategy with the convective eigenvalue problem formulations to tackle the problem on transient internal conjugate convection-conduction heat transfer. The data obtained for the three representative situations with Pe = 1, 10, and 100 demonstrate a remarkable convergence improvement in comparison with the classical solution via the diffusive eigenvalue problem. Despite the lack of formal proof regarding the completeness of the set of eigenfunctions, the results presented in this work bring enough evidence on the validity of the expansion constructed upon the nonclassical eigenvalue problem proposed.

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TABLE 3. Convergence Behavior of the Calculated Temperatures at XPe = 0.05, τ = 0.01, and Pe = 100

N	$Y = 0.2$	$Y = 0.3$	$Y = 0.4$	$Y = 0.6$	$Y = 0.7$	$Y = 0.8$
Convective eigenvalue problem						
3	0.003741	0.021366	0.084100	0.296500	0.295620	0.280568
6	0.008122	0.029257	0.087402	0.289878	0.316661	0.288770
9	0.008837	0.028693	0.086863	0.292425	0.314506	0.290178
12	0.008819	0.028761	0.087426	0.292684	0.313462	0.290293
15	0.008837	0.028817	0.087474	0.292616	0.313595	0.290154
18	0.008853	0.028836	0.087467	0.292643	0.313572	0.290172
21	0.008854	0.028835	0.087468	0.292644	0.313575	0.290177
24	0.008853	0.028835	0.087469	0.292644	0.313574	0.290177
Diffusive eigenvalue problem						
3	0.035549	0.077739	0.124173	0.288812	0.332060	0.283718
6	0.021022	0.061520	0.128831	0.293968	0.307020	0.280323
9	0.024414	0.056592	0.124450	0.278804	0.300961	0.298172
12	0.006758	0.036123	0.102760	0.281234	0.324817	0.295941
15	0.002775	0.027560	0.095919	0.295227	0.315820	0.286695
18	0.007258	0.024736	0.092961	0.296046	0.312206	0.291148
21	0.006506	0.026208	0.091243	0.295428	0.312691	0.290913
24	0.006978	0.026460	0.090371	0.294088	0.313559	0.290248
27	0.006721	0.026742	0.090026	0.293208	0.314674	0.289590
30	0.008928	0.028412	0.090795	0.292538	0.315290	0.288825
33	0.009008	0.029836	0.089682	0.293304	0.313915	0.290327
36	0.009399	0.029672	0.089438	0.292920	0.314083	0.290612
39	0.009433	0.029942	0.089093	0.293266	0.313421	0.290605
42	0.009430	0.030087	0.088898	0.293044	0.313782	0.290248
45	0.009425	0.030036	0.088944	0.292863	0.313666	0.290305
48	0.009488	0.029963	0.088763	0.292726	0.313679	0.290222
51	0.009410	0.029923	0.088763	0.292769	0.313782	0.290179
54	0.009432	0.029918	0.088709	0.292690	0.313659	0.290228
57	0.008790	0.029345	0.088379	0.293132	0.313380	0.290570
60	0.008639	0.029085	0.088285	0.292984	0.313603	0.290320
63	0.008734	0.029075	0.088248	0.292995	0.313702	0.290264
66	0.008789	0.029104	0.088261	0.292976	0.313672	0.290253
69	0.008775	0.029121	0.088341	0.293011	0.313686	0.290256

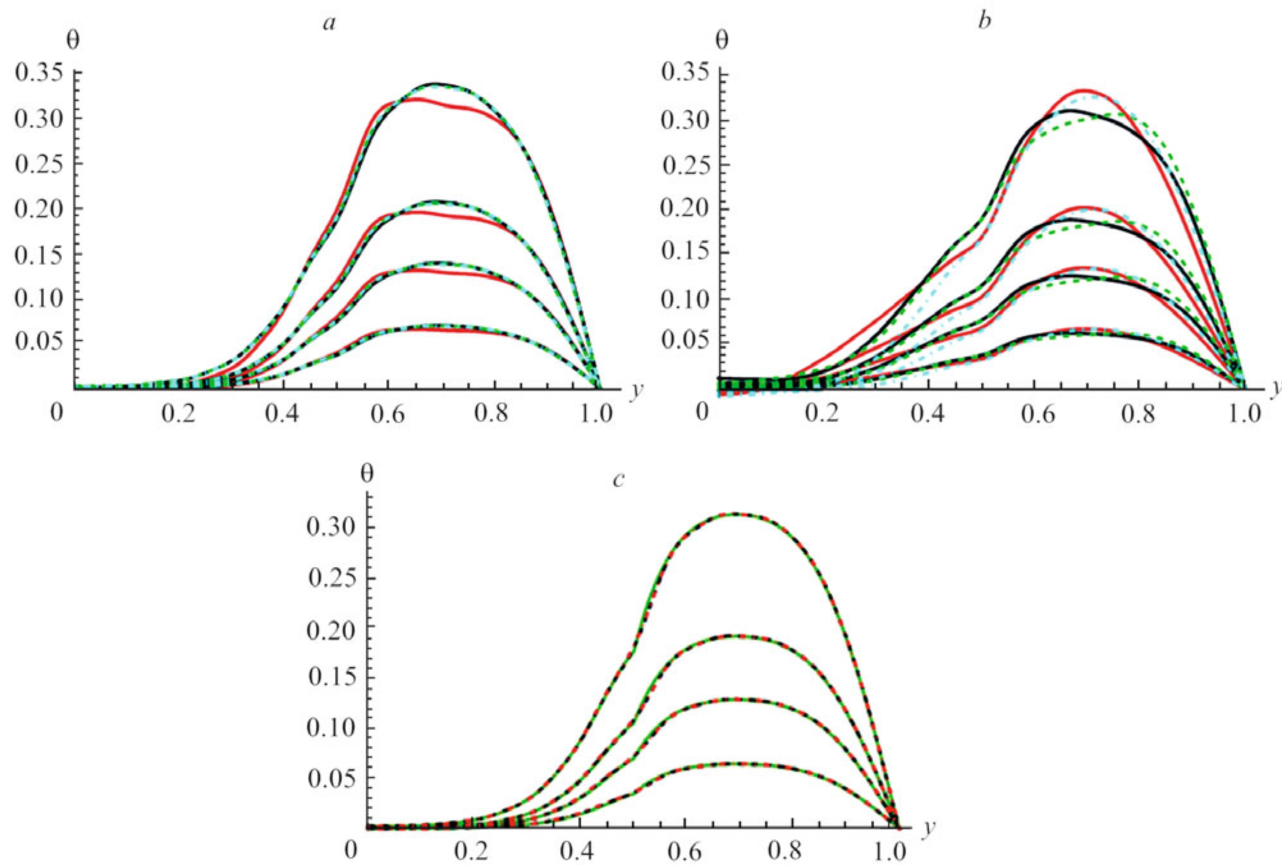

Fig. 4. Transversal temperature profiles calculated employing the convective eigenvalue problem (a) and the diffusive eigenvalue problem (b) with $N = 3$ (red lines), 6 (black lines), 9 (green dashed lines) and 12 (cyan dash-dotted lines) and employing the Comsol Multiphysics (green lines), the convective eigenvalue problem with $N = 9$ (black dashed lines), and the diffusive eigenvalue problem with $N = 69$ (red dash-dotted lines) at Pe =100.

NOTATION

K, thermal conductivity; T, temperature; t, time variable; u, x component of the velocity field; w, volumetric heat capacity; x, longitudinal coordinate; y, transverse coordinate; α, thermal diffusivity; β, eigenvalue corresponding to ζ; ζ, eigenfunction of the convective eigenvalue problem; θ, dimensionless temperature; λ, eigenvalue corresponding to Ω; μ, eigenvalue corresponding to ψ; τ, dimensionless time variable; ψ, eigenfunction of the diffusive eigenvalue problem; Ω, eigenfunction of the auxiliary eigenvalue problem. Subscripts: f, fluid; s, solid wall.

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