As the critical level is approached,

$$\frac{\partial \sigma}{\partial k} \to U + A\,, \quad \frac{\partial \sigma}{\partial l} \to 0\,. \tag{4.20ab}$$

Therefore the wave group is effectively captured in this neighborhood and constrained thereafter to propagate along the mean flow.

5. Discussions and conclusions

The propagation of Alfvén waves has been studied in an isothermal atmosphere when the displacement current is included in this analysis. It is shown that the governing wave equation is singular at heights where the flow velocity U assumes special values such that the Doppler shifted frequency $\omega = 0$, $\pm\omega_a$, $\pm\lambda$ and $\pm\delta$. When the displacement current is neglected, the singularities correspond to $\omega = 0$ and $\pm\omega_a$. Thus the effect of displacement current is to increase the number of singularities of the equation, and, in turn, is to reduce the gross attenuation of the wave across the critical levels.

In the limit $c \to \infty$, the critical values $\pm\lambda$ and $\pm\delta$ coincide with the critical values $\pm\omega_a$.

Since the wave action flux, $\tilde{M}$ is constant everywhere except at the critical levels, we found it as an appropriate measure of the magnitude of the waves. Using this wave action flux as a measure of intensity of the wave, the attenuation of waves at the critical levels is examined.

It is shown that the attenuation that a wave suffers on passage through the critical levels $\zeta = 0$ (that is, away from all the above seven critical levels) in an isothermal medium is less than the attenuation that a wave suffers on passage through the critical levels in a fluid in which the displacement current is neglected. This result is confirmed using the group velocity approach. Finally, it is confirmed that the effect of the displacement current is to increase the number of critical layers and in turn, is to reduce the gross attenuation of waves across the critical levels.

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On the Matching Conditions for a Two-Deck Compressible Turbulent Boundary Layer Model

I. Introduction

Over the past forty years the problem of compressible turbulent boundary layers has been the subject of many theoretical and experimental studies. The difficulty of treating both turbulence and compressibility, and the success in describing incompressible turbulent boundary layers, has motivated a number of authors to look for features of the incompressible flow which are preserved in compressible flow. These identified common features, together with the procedure for incompressible flow, and adequate modifications, can then be used to try to develop a theory for compressible flow.

Recently, some authors (MELLOR [12], YAJNIK [15] and others) have developed formal asymptotic theories for incompressible turbulent boundary layers, employing the method of matched asymptotic expansions together with asymptotic hypotheses describing the order of the various terms in the equations of mean motion. The theories consider the entire fluid region and avoid similarity, dimensional or eddy viscosity arguments. The main idea underlying the theories is that there are two different layers whose properties and thickness can be described by their respectives length scales.

Using the same idea, AFZAL [2] attempted to extend the asymptotic theories developed for incompressible flow to compressible flow. He formulated a higher order theory for compressible turbulent boundary layer flow of a perfect gas with constant specific heats when $(\gamma - 1)\, M_\infty^2$ and molecular Prandtl number are of order unity; he worked with an underdetermined system of equations of mean motion and showed that the structure of the solution is the same as for incompressible flow, i.e., two layers are required. However, several authors, MELNIK and GROSSMAN [13], ADAMSON and FEO [1], LIOU and ADAMSON [10], pointed out that although AFZAL had presented the solution by limit-function expansions, he had worked with an underdetermined system of equations and had not showed that the wall and the defect solutions did, in fact, match. Thus, according to their analysis the fact that the density varied by order unity from wall to free stream values across the boundary layer caused difficulties which could not be solved by AFZAL's solution. It is of interest to note that the derivation of this result strongly depends on AFZAL's assumed form of the asymptotic expansions, and that no systematic study about the domains of validity of the solutions was made. Thus, those analyses cannot be seen as conclusive proof that the defect and wall layers do not match.

The aim of this work is to carry out a study of the domains of validity of the defect and wall layer solutions using the intermediate variable technique. This technique was introduced by KAPLUN in the mid fifties and reviewed by LAGERSTROM and CASTEN in the early seventies. The intermediate variable technique provides a more rigorous mathematical treatment to deal with problems where perturbation techniques can be applied. In this theory, heuristic ideas are used to develop a systematic method of obtaining domains of validity of asymptotic expansions by a study of the corresponding equations. The theory emphasizes the idea of characterizing expansions by their domains of validity, providing not only a more basic understanding for the matching but also a deeper insight into the construction of the expansions. This is illustrated in the work of LAGERSTROM and CASTEN [8], which also shows how it can be applied to different problems.

II. Equations of mean motion and asymptotic hypotheses

In this section we introduce the two-dimensional Navier-Stokes equations of mean motion for a compressible fluid using the mass-weighted-averaging procedure and follow AFZAL's analysis of KISTLER's data to estimate the order of magnitude of the turbulent fluctuations of velocity, pressure, temperature and density.

The continuity, momentum, energy and state equations obtained by mass-weighted averaging can be written in the following non-dimensional form:

a) Continuity

$$\frac{\partial}{\partial x_j}\left(\bar{\varrho}\tilde{u}_j\right) = 0 \tag{1}$$

b) Momentum

$$\frac{\partial}{\partial x_j}\left(\bar{\varrho}\tilde{u}_i\tilde{u}_j\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(-\overline{\varrho u_i' u_j'} + R^{-1}\overline{\tau_{ij}}\right) \tag{2}$$

c) Energy

$$\frac{\partial}{\partial x_j}(\bar{\varrho}\tilde{t}\tilde{u}_j) = D\tilde{u}_j\frac{\partial \bar{p}}{\partial x_j} + D\overline{u'_j\frac{\partial p}{\partial x_j}} + \\ + \frac{\partial}{\partial x_j}(R^{-1}\sigma^{-1}\bar{q}_j - \overline{\varrho t' u'_j}) + R^{-1}D\overline{\tau_{ij}\frac{\partial u_i}{\partial x_j}}, \tag{3}$$

d) State

$$\bar{p} = \frac{\gamma - 1}{\gamma}\frac{1}{D}\bar{\varrho}\tilde{t}, \tag{4}$$

where the stress tensor τ_{ij} and the heat flux tensor q are given respectively by

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial u_l}{\partial x_l} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{5}$$

and

$$q_j = -K\frac{\partial t}{\partial x_j}. \tag{6}$$

In these equations, x, u, p, ϱ, and t have their classical meaning, λ is the bulk viscosity $(= -2/3\mu)$, μ the dynamic viscosity, K the thermal viscosity, and δ_{ij} the Kronecker delta, having the value 1 for $i = j$ and 0 for $i \neq j$. All lengths are non-dimensionalized by a typical body dimension L, velocities by the characteristic reference velocity U_∞, pressure by $\varrho_\infty U_\infty^2$, temperature by T_∞, density by ϱ_∞ and viscosity by μ_∞. The quantity $R = U_\infty\varrho_\infty L/\mu_\infty$ is the characteristic Reynolds number, M_∞ is a characteristic Mach number, $D = (\gamma - 1)M_\infty^2$ is the compressibility factor, σ is the molecular Prandtl number and γ is the ratio of specific heats. The superscript denotes turbulent fluctuations, the bar denotes conventional time averaging and the tilde denotes mass-weighted averaging. A summation is understood for repeated indices. For further details concerning these equations the reader is referred to either CEBECI and SMITH [4] or LIEPMANN and ROSHKO [9].

Based on the measurements of KISTLER [6], AFZAL concluded that the fluctuation u' can be assumed to be of the order of the friction velocity divided by the mean free stream velocity, i.e.,

$$u' = O\left(\frac{u_\tau}{U_\infty}\right) = O\left(\frac{1}{U_\infty}\sqrt{\frac{\tau_w}{\varrho_w}}\right) = O(\varepsilon). \tag{7}$$

For the static pressure, density and temperature fluctuations, further measurements of KISTLER and CHEN [7] and of MORKOVIN [14] have shown that (a) u', ϱ' and t' have the same order of magnitude and (b) the root-mean-squate value of p' is proportional to u'.

The notation for turbulent Reynolds terms employed here will be the same as presented by AFZAL, i.e., second order correlations of the type $\overline{a'b'}$ are denoted by τ_{ab} and third-order correlations of the type $\overline{a'b'c'}$ by τ_{abc}. Thus a correlation of the type $\overline{-b'\partial a'/\partial y}$ is denoted by $-\overline{a'_y b'}$. In view of the above remarks, the scales of fluctuations can be written without loss of generality as

$$O(u') = O(v') = O(\varrho') = O(t') = O(\varepsilon) \tag{8}$$

$$O(p') = \varepsilon^2. \tag{9}$$

III. Analysis of the domains of validity of the asymptotic expansions

In this section we apply the intermediate variable technique to the problem of compressible turbulent boundary layers. This requires the knowledge of the concepts of domain of validity of asymptotic expansions, overlap, formal validity of equations, limit process, etc. Most of these concepts lie in ideas of the late SAUL KAPLUN and his co-workers, and have been presented with particular clarity in an article of LAGERSTROM and CASTEN [8]. Basic concepts such as those of asymptotic expansion, order classes, uniform convergence on a function class and uniform validity on a function class will not be introduced here since they have been extensively treated in the literature. The order to which a function $\eta(\varepsilon)$ belongs will be denoted by ord η.

Before we proceed, let us remember that the parameters σ and D are assumed to be order unity, and that the boundary layer has three length scales [AFZAL, 2]: an inviscid scale L, an outer scale $\delta = \varepsilon L$, and an inner scale $\hat{\delta} = \nu/u_\tau = \hat{\varepsilon}\delta$. The ratio of the outer to the inviscid length scale is $\delta/L = \varepsilon$ and of the inner to the outer scale is

$$\hat{\delta}/\delta = \nu/\delta u_\tau = \hat{\varepsilon}. \tag{10}$$

Moreover, parameters ε and $\hat{\varepsilon}$ are related by the well known turbulent skin friction equation which yields [MELLOR, 12]

$$\varepsilon = O((\ln\hat{\varepsilon})^{-1}), \quad \text{as} \quad \varepsilon \to 0. \tag{11}$$

Our objective is to obtain an approximation for the system of equations (1—6) which is uniformly valid in a given domain. For this reason, it is necessary to define the concept of uniform domain of validity for such approximation. Here, our first difficulty occurs, since, although we can formalize clearly the concept of uniform validity on a function class, i.e., when solutions are close, the basic difficulty of deciding when equations are close still remains unanswered.

Therefore in order to introduce KAPLUN's matching principle it is firstly necessary to define η-limit and the domain of validity of equations. The η-limit of an equation $E(x, y; \varepsilon)$ is defined as follows. Let the intermediate variable y_η be

$$\eta(\varepsilon)\, y_\eta = y, \tag{12}$$

where, as indicated, η is a function of ε. Then, the η-limit of $E(x, y; \varepsilon)$ is

$$\lim\nolimits_\eta E(x, y; \varepsilon) = \lim E(x, \eta(\varepsilon)\, y_\eta; \varepsilon) \quad \text{as} \quad \varepsilon \to 0 \tag{13}$$

with y_η fixed.

Note that $\eta(\varepsilon)$, as defined by equation (12), is in fact a stretching function. If we substitute (12) into the motion equation and arbitrarily vary the order of magnitude of $\eta(\varepsilon)$, we can then use the definition of η-lim to study the effect on the motion equations. This will reveal the importance of the various terms of the equation of motion in the different flow regions. Now, if we introduce (12) into the system (1—6), each term will have a formal order in ε depending on the assumptions in section II and on expressions (10) and (11). The derivatives and y_η are formally considered to be of order unity. For example, we note that the terms $(\bar{\varrho}\bar{\tau}_{uv})_y$ and $\bar{\mu}\tilde{u}_{yy}$, in equation (2), are formally of order ε^2/η and $\varepsilon^2\hat{\varepsilon}/\eta$ respectively. Passing the η-limit in the equations resulting from (2—3), we find the following formal limits depending on different order of magnitude of η:

x-momentum equation:

ord $\eta > 1$:	$(\bar{\varrho}\tilde{u})\,\tilde{u}_x + (\bar{\varrho}\tilde{v})\,\tilde{u}_y + \bar{p}_x = 0$,	(14a)
ord $\eta = 1$:	$(\bar{\varrho}\tilde{u})\,\tilde{u}_x + (\bar{\varrho}\tilde{v})\,\tilde{u}_y + \bar{p}_x = 0$,	(14b)
ord $\varepsilon <$ ord $\eta < 1$:	$(\bar{\varrho}\tilde{u})\,\tilde{u}_x + (\bar{\varrho}\tilde{v})\,\tilde{u}_y + \bar{p}_x = 0$,	(14c)
ord $\eta =$ ord ε^2:	$(\bar{\varrho}\tilde{u})\,\tilde{u}_x + (\bar{\varrho}\tilde{v})\,\tilde{u}_y + \bar{p}_x = (\bar{\varrho}\bar{\tau}_{uv})_y$,	(14d)
ord $\varepsilon^3 <$ ord $\eta <$ ord ε^2:	$(\bar{\varrho}\bar{\tau}_{uv})_y = 0$,	(14e)
ord $\hat{\varepsilon} <$ ord $\eta <$ ord ε^3:	$(\bar{\varrho}\bar{\tau}_{uv})_y = 0$,	(14f)
ord $\eta =$ ord $\hat{\varepsilon}$:	$(\bar{\mu}\tilde{u}_y)_y + (\bar{\varrho}\bar{\tau}_{uv})_y = 0$,	(14g)
ord $\eta <$ ord $\hat{\varepsilon}$:	$(\bar{\mu}\tilde{u}_y)_y = 0$;	(14h)

energy equation:

ord $\eta > 1$:	$\bar{\varrho}\tilde{u}\tilde{t}_x + \bar{\varrho}\tilde{v}\tilde{t}_y - D(\tilde{u}\bar{p}_x + \tilde{v}\bar{p}_y) = 0$,	(15a)
ord $\eta = 1$:	$\bar{\varrho}\tilde{u}\tilde{t}_x + \bar{\varrho}\tilde{v}\tilde{t}_y - D(\tilde{u}\bar{p}_x + \tilde{v}\bar{p}_y) = 0$,	(15b)
ord $\varepsilon^2 <$ ord $\eta < 1$:	$\bar{\varrho}\tilde{u}\tilde{t}_x + \bar{\varrho}\tilde{v}\tilde{t}_y - D(\tilde{u}\bar{p}_x + \tilde{v}\bar{p}_y) = 0$,	(15c)
ord $\eta =$ ord ε^2:	$\bar{\varrho}\tilde{u}\tilde{t}_x + \bar{\varrho}\tilde{v}\tilde{t}_y - D(\tilde{u}\bar{p}_x + \tilde{v}\bar{p}_y) = (\bar{\varrho}\bar{\tau}_{vt})_y + D(\bar{\tau}_{vp})_y$,	(15d)
ord $\varepsilon^3 <$ ord $\eta <$ ord ε^2:	$(\bar{\varrho}\bar{\tau}_{vt})_y + D(\bar{\tau}_{vp})_y = 0$,	(15e)
ord $\hat{\varepsilon} <$ ord $\eta <$ ord ε^3:	$(\bar{\varrho}\bar{\tau}_{vt})_y + D(\bar{\tau}_{vp})_y = 0$,	(15f)
ord $\eta =$ ord $\hat{\varepsilon}$:	$\sigma^{-1}(\bar{\mu}\tilde{t}_y)_y + (\bar{\varrho}\bar{\tau}_{vt})_y + D(\bar{\tau}_{vp})_y = 0$,	(15g)
ord $\eta <$ ord $\hat{\varepsilon}$:	$\sigma^{-1}(\bar{\mu}\tilde{t}_y)_y + 0$,	(15h)

The two sets of equations, (14a—h) and (15a—h), are the two important ones. Indeed, the continuity equation, y-momentum equation and the state equation result in trivial sets of equations. For instance, the y-momentum equation, whatever is the order of η, results in

$$\bar{p}_y = 0. \tag{16}$$

We also note that if $\lim_\eta$ applied to equations (2—3) yields (14e—15e), then the formal η-limit of (14d—15d) also yields

(14e—15e). Thus according to LAGERSTROM and CASTEN, equations (14d—15d) are "rich enough" to contain (14e—15e). The two definitions below are due to LAGERSTROM and CASTEN:

Definition I. If E is an equation and $\lim_{\eta_1} E = E_1$, $\lim_{\eta_2} E = E_2$, and also $\lim_{\eta_2} E_1 = E_2$, we say that E_1 contains E_2 (relative to E).

Definition II. The formal domain of validity of an equation F, relative to the full equation E, is ord η such that $\lim_\eta E$ is either F or an equation contained in F.

These definitions imply that the formal domain of validity of (14d) and (15d) is given by

$$D_0 = \{\eta/\text{ord}\ \eta > \text{ord}\ \hat{\varepsilon}\}\,, \tag{17}$$

and that of (14g) and (15g) is given by

$$D_i = \{\eta/\text{ord}\ \eta < \text{ord}\ \varepsilon^2\}\,. \tag{18}$$

Equations d's and g's are called the principal equations. They are, in fact, the important equations. One can observe that equations d's contain equations a's, b's, c's, e's and f's; equations g's contain equations e's, f's and h's but neither of them is contained in the other.

Principle. If f is a solution of an equation E, and E^* is an approximate equation, then there exists a solution f^* of E^* whose actual domain of validity (as an approximation to f) includes the formal domain of validity of E^* (as an approximation to E).

In the LAGERSTROM and CASTEN article, important modifications of this principle are discussed since it ceases to be valid when small terms have large integrated effects. Fortunately, terms such as $\varepsilon \ln \varepsilon$, which appeared in the solution of LAGERSTROM and CASTEN's second mode equation, are not expected to occur here. These terms provoke an effect of larger magnitude than the order formally indicated which cannot be compensated by the lower order terms leading to the phenomenon of integrated effects and resulting switchback. So, according to the above remarks, it seems that KAPLUN's principle can be applied to the problem defined by equations (1—6). These results yields, together with results (17—18), that the domain of overlap is given by

$$D = D_0 \cap D_i = \{\eta/\text{ord}\ \hat{\varepsilon} < \text{ord}\ \eta < \text{ord}\ \varepsilon^2\}\,. \tag{19}$$

Equations (14 a—h) illustrate the well known balance among inertia, pressure, Reynolds stress and viscous stress in the flow. In fact, equation (14h) shows that very close the wall the effects due to the viscous shear stress are the dominating effects; as the distance from the wall increases, equation (14g) shows that the Reynolds shear stress becomes larger and balances the viscous shear stress, both effects being of lower order than the pressure gradient effects. Further away, the overlap region is reached, and equations (14 e—f) show that the Reynolds stress effects are the dominating effects. Above this region, equation (14d) shows that the inertia, pressure and Reynolds stress are of equal importance. Finally, in the outermost layer, comprising most of the boundary layer, equations (14 a—c) show that the inertia and pressure effects are the dominating effects.

Of course, a similar type of analysis applied to the energy equation emphasizes, near the wall, the conduction terms and, further away from the wall, the turbulent terms.

IV. Asymptotic analysis

In this section, we apply the matched asymptotic expansion method to the problem of compressible, turbulent, boundary layers to show how solutions for the defect and wall layers can be obtained which do match in an overlap domain. Since most of the analysis concerning the obtaining of the asymptotic expansion is conventional, only a few comments about it will be made here. The bars and tildes will be omitted in all subsequent expressions.

Defect layer

For this region we write

$$u(x, y; R) = u_1(x, y) + \varepsilon u_2(x, y) + \varepsilon^2 u_3(x, y) + \dots\,, \tag{20a}$$

$$v = \varepsilon v_1(x, y) + \varepsilon^2 v_2(x, y) + \varepsilon^3 v_3(x, y) + \dots\,, \tag{20b}$$

$$p = p_1(x, y) + \varepsilon p_2(x, y) + \varepsilon^2 p_3(x, y) + \dots\,, \tag{20c}$$

$$\varrho = \varrho_1(x, y) + \varepsilon \varrho_2(x, y) + \varepsilon^2 \varrho_3(x, y) + \dots\,, \tag{20d}$$

$$t = t_1(x, y) + \varepsilon t_2(x, y) + \varepsilon^2 t_3(x, y) + \dots\,, \tag{20e}$$

$$\tau_{ab} = \varepsilon^2 \tau_{ab_1}(x, y) + \varepsilon^3 \tau_{ab_2}(x, y) + \varepsilon^4 \tau_{ab_3}(x, y) + \dots\,, \tag{20f}$$

$$\tau_{abc} = \varepsilon^3 \tau_{abc_1}(x, y) + \varepsilon^4 \tau_{abc_2}(x, y) + \varepsilon^5 \tau_{abc_3}(x, y) + \dots\,. \tag{20g}$$

where $y = Y/\delta(R)$ is the stretched defect layer variable.

Introduction of these expressions into the equations of mean motion and collection of coefficients of various powers of ε give the equations for the successive approximations. These equations are, except for a few differences arising from distinct averaging procedures, the same as those obtained by AFZAL. They show that the pressure is constant across the outer layer to order ε and that viscous and heat-conduction terms do not occur in the outer layer. From the matching between the inviscid and outer expansions (see AFZAL), it follows that

$$\left.\begin{array}{lll} u_1(x, y) = U_1(x, 0)\,, & p_1(x, y) = P_1(x, 0)\,, & \\ \varrho_1(x, y) = \mathrm{P}_1(x, 0)\,, & t_1(x, y) = T_1(x, 0)\,, & y \to \infty\,, \\ \tau_{ab_1}(x, y) = \mathrm{T}_{ab_1}(x, 0)\,, & \tau_{abc_1}(x, y) = \mathrm{T}_{abc_1}(x, y)\,. & \end{array}\right\} \tag{21}$$

Here, capital letters are used to denote terms valid in the inviscid region and, in order to follow AFZAL's notation, the quantities $U_1(x, 0)$, $P_1(x, 0)$, $\mathrm{P}_1(x, 0)$ and $T_1(x, 0)$ are denoted by U_{10}, P_{10}, P_{10} and T_{10} respectively. The solution of the lowest order equation which satisfies the matching conditions is

$$\left.\begin{array}{ll} u_1 = U_{10}\,, & \varrho_1 = \mathrm{P}_{10}\,, \\ v_1 = [Q - (\mathrm{P}_{10} U_{10x})]/\mathrm{P}_{10}\,, & \\ p_1 = P_{10}\,, & t_1 = T_{10}\,. \end{array}\right\} \tag{22}$$

where Q is a constant of integration to be determined à posteriori.

Wall layer

To describe the flow behavior near the wall we introduce the inner variable

$$\hat{y} = y\hat{\varepsilon} = y\varepsilon R_w\,,$$

where $R_w = U_\infty L/\nu_{\mathrm{w}}$, ν_{w} = local kinematic viscosity at the wall. The appropriate asymptotic expansions for this region are

$$\hat{u} = \varepsilon \hat{u}_1(x, \hat{y}) + \varepsilon^2 \hat{u}_2(x, \hat{y}) + \dots\,, \tag{23a}$$

$$\hat{v} = \varepsilon\hat{\varepsilon}[\hat{v}_1(x, \hat{y}) + \varepsilon \hat{v}_2(x, \hat{y}) + \dots]\,, \tag{23b}$$

$$\hat{p} = \hat{p}_1(x, \hat{y}) + \varepsilon \hat{p}_2(x, \hat{y}) + \dots\,, \tag{23c}$$

$$\hat{\varrho} = \hat{\varrho}_1(x, \hat{y}) + \varepsilon \hat{\varrho}_2(x, \hat{y}) + \dots\,, \tag{23d}$$

$$\hat{t} = \hat{t}_1(x, \hat{y}) + \varepsilon \hat{t}_2(x, \hat{y}) + \dots\,, \tag{23e}$$

$$\hat{\tau}_{ab} = \varepsilon^2 \hat{\tau}_{ab_1}(x, \hat{y}) + \varepsilon^3 \hat{\tau}_{ab_2}(x, \hat{y}) + \dots\,, \tag{23f}$$

$$\hat{\tau}_{abc} = \varepsilon^3 \hat{\tau}_{abc_1}(x, \hat{y}) + \varepsilon^4 \hat{\tau}_{abc_2}(x, \hat{y}) + \dots\,. \tag{23g}$$

An asymptotic expansion for the viscosity is derived expanding μ in a Taylor series around t_1, that is,

$$\mu = \mu(\hat{t}_1) + \varepsilon \mu_t(\hat{t}_1)\,\hat{t}_2 + O(\varepsilon^2)\,, \tag{24}$$

$$\text{where, } \mu_t = \left.\frac{\partial \mu}{\partial t}\right|_{\hat{t}=\hat{t}_1}.$$

The result of inserting (23) and (24) into (1—6) and collecting terms of same order is:

to lowest order terms:

$$(\hat{\varrho}_1 \hat{u}_{j_1})_{x_j} = 0\,, \tag{25a}$$

$$(\hat{\varrho}_1 \hat{\tau}_{uv_1})_{\hat{y}} + \frac{\mu(\hat{t}_1)}{\nu_{\mathrm{w}}}\,\hat{u}_{1_{\hat{y}\hat{y}}} = 0\,, \tag{25b}$$

$$\hat{p}_{1_{\hat{y}}} = 0\,, \qquad \hat{t}_{1_{\hat{y}\hat{y}}} = 0\,, \tag{25 c, 25 d}$$

$$\hat{p}_1 = \frac{\gamma - 1}{\gamma}\,\frac{1}{D}\,\hat{\varrho}_1 \hat{t}_1; \tag{25e}$$

to first order terms:

$$(\hat{\varrho}_1 \hat{u}_{j_2})_{x_j} + (\hat{\varrho}_2 \hat{u}_{j_1})_{x_j} = 0\,, \tag{26a}$$

$$(\hat{\varrho}_1 \hat{\tau}_{uv_2})_{\hat{y}} + (\hat{\varrho}_2 \hat{\tau}_{uv_1})_{\hat{y}} + \hat{\tau}_{\varrho uv_{1_{\hat{y}}}} + \frac{1}{\nu_{\mathrm{w}}}\,[\mu(\hat{t}_1)\,\hat{u}_{2_{\hat{y}\hat{y}}} + \mu_t(\hat{t}_1)\,\hat{t}_2 \hat{u}_{1_{\hat{y}\hat{y}}}] = 0\,, \tag{26b}$$

$$p_{2_{\hat{y}}} = 0\,, \tag{26c}$$

$$\frac{1}{\sigma \nu_{\mathrm{w}}}\,[\mu(\hat{t}_1)\,\hat{t}_{2_{\hat{y}\hat{y}}}] + (\hat{\varrho}_1 \hat{\tau}_{tv_1})_{\hat{y}} = 0\,, \tag{26d}$$

$$\hat{p}_2 = \frac{\gamma - 1}{\gamma}\,\frac{1}{D}\,(\hat{\varrho}_1 \hat{t}_2 + \hat{\varrho}_2 \hat{t}_1)\,. \tag{26e}$$

Equations 25 (c—e) and the matching conditions show that pressure, temperature and density are constant across the wall layer up to the lowest order terms. Hence, we can write

$$\left.\begin{aligned} p_1 &= p_1(x, y) = P_{10}\,, \\ \hat{t}_1 &= t_{\mathrm{W}}\,, \qquad \hat{\varrho}_1 = \varrho_{\mathrm{W}}\,. \end{aligned}\right\} \tag{27}$$

Furthermore, for an impermeable wall, the matching of the pressure in the wall, outer and inviscid regions gives

$$\hat{p}_2(x, \hat{y}) = p_2(x, y) = P_2(x, Y) = 0\,, \tag{28}$$

The velocity profiles for the defect and wall layers are given by

$$u = U_{10} + \varepsilon(a_{11} \ln y + a_{12}) + \varepsilon^2(a_{21} \ln^2 y + {} + a_{22} \ln y + a_{23}) + O(\varepsilon^3) \tag{29a}$$

and

$$\hat{u} = \varepsilon(\hat{a}_{11} \ln \hat{y} + \hat{a}_{12}) + \varepsilon^2(\hat{a}_{21} \ln^2 \hat{y} + \hat{a}_{22} \ln \hat{y} + \hat{a}_{23}) + O(\varepsilon^3) \tag{29b}$$

where the a's represent arbitrary functions of y.

The presence of first order logarithmic terms in the above expressions is justified by equation (25b) and the matching conditions for the defect and wall layers [AFZAL, 2]. The second-order bilogarithmic terms occur due to expression (26b). Indeed, equations (26d) and (26e) yield that the first-order density term is logarithmic; substitution of this term and the solution of (25b) into (26b) results in a solution which contains bilogarithmic terms. Thus bilogarithmic terms occur only in compressible flow.

An analysis similar to that for the velocity profiles carried out for the temperature profiles, suggests that

$$t = t_{10} + \varepsilon(b_{11} \ln y + b_{12}) + {} + \varepsilon^2(b_{21} \ln^2 y + b_{22} \ln y + b_{23}) + O(\varepsilon^3)\,, \tag{30a}$$

and

$$\hat{t} = t_w + \varepsilon(\hat{b}_{11} \ln \hat{y} + \hat{b}_{12}) + {} + \varepsilon^2(\hat{b}_{21} \ln^2 \hat{y} + \hat{b}_{22} \ln \hat{y} + \hat{b}_{23}) + O(\varepsilon^3)\,, \tag{30b}$$

where the b's are arbitrary functions of y.

On substituting the above profiles into the equation of state and taking into account the conditions imposed by equations (27) and (28), it follows that the density profiles are given by

$$\varrho = \mathrm{P}_{10} - \varepsilon \frac{\mathrm{P}_{10}}{T_{10}} (b_{11} \ln y + b_{12}) + {} + \varepsilon^2(c_{21} \ln^2 y + c_{22} \ln y + c_{23}) + O(\varepsilon^3)\,, \tag{31a}$$

and

$$\hat{\varrho} = \varrho_{\mathrm{W}} - \varepsilon \frac{\varrho_{\mathrm{W}}}{t_{\mathrm{W}}} (\hat{b}_{11} \ln \hat{y} + \hat{b}_{12}) + \varepsilon^2(\hat{c}_{21} \ln^2 \hat{y} + \hat{c}_{22} \ln \hat{y} + \hat{c}_{23}) + {} + O(\varepsilon^3)\,. \tag{31b}$$

where the c's are arbitrary functions of y.

Now, whatever techniques are used, matching is always carried out by comparing the outer expansion for small y with the inner expansion for large $\hat{y}$; the limit of the outer expansions as $y \to 0$, and the limit of the inner expansion as $\hat{y} \to \infty$, are shown in Table 1.

In this Table, each row defines a set of three equations to be satisfied, the tangential velocity, the temperature and the density matching condition equations. It is interesting to note that as $\hat{y} \to \infty$, the logarithmic terms give contributions of lower order of magnitude than the order formally indicated; this fact is fundamental for a successful match. Indeed, if the bilogarithmic terms are dropped, the matching condition between the velocity profiles yields

$$U_{10} = \varepsilon \ln \hat{\varepsilon}\, \hat{a}_{11} + \varepsilon \hat{a}_{12} - \varepsilon a_{12} \tag{32a}$$

so that $\varepsilon \ln \hat{\varepsilon} = O(1)$ as $\varepsilon \to 0$.

Equation (32) is the classical skin-friction equation obtained by MELLOR [12]. Thus the temperature and density matching conditions become

$$t_{\mathrm{W}} + \varepsilon \ln \hat{\varepsilon} \hat{b}_{11} + \varepsilon \hat{b}_{12} = T_{10} + \varepsilon b_{12} \tag{33a}$$

and

$$\varrho_{\mathrm{W}} + \varepsilon \ln \hat{\varepsilon} \frac{\varrho_{\mathrm{W}}}{t_{\mathrm{W}}} \hat{b}_{11} + \varepsilon \frac{\varrho_{\mathrm{W}}}{t_{\mathrm{W}}} \hat{b}_{12} = \mathrm{P}_{10} + \varepsilon \frac{\mathrm{P}_{10}}{T_{10}} b_{12}\,. \tag{34a}$$

The matching difficulties can now be clearly seen from equations (33a) and (34a), that is, as $\varepsilon \to 0$ parameter $\hat{b}_{11}$ alone is not capable of satisfying both equations. Hence, in order to make the matching feasible, one must consider the second-order bilogarithmic terms. Then the matching conditions given by equations (32a—34a) become

$$U_{10} = \varepsilon \ln \hat{\varepsilon} \hat{a}_{11} + \varepsilon \hat{a}_{12} - \varepsilon a_{12} + {} + \varepsilon^2 \ln^2 \hat{\varepsilon} \hat{a}_{21} + \varepsilon^2 \ln \hat{\varepsilon} \hat{a}_{21} + \varepsilon^2 a_{23} - \varepsilon^2 \hat{a}_{23}\,, \tag{32b}$$

$$t_{\mathrm{W}} - T_{10} = -\varepsilon \ln \hat{\varepsilon} \hat{b}_{11} - \varepsilon \hat{b}_{12} + \varepsilon b_{12} - {} - \varepsilon^2 \ln^2 \hat{\varepsilon} \hat{b}_{21} - \varepsilon^2 \ln \hat{\varepsilon} \hat{b}_{22} - \varepsilon^2 \hat{b}_{23} + \varepsilon^2 b_{23}\,, \tag{33b}$$

$$\mathrm{P}_{10} - \varrho_{\mathrm{W}} = \varepsilon \ln \hat{\varepsilon} \frac{\varrho_{\mathrm{W}}}{t_{\mathrm{W}}} \hat{b}_{11} + \varepsilon \frac{\varrho_{\mathrm{W}}}{t_{\mathrm{W}}} \hat{b}_{12} - \varepsilon \frac{\mathrm{P}_{10}}{T_{10}} b_{12} + \varepsilon^2 \ln^2 \hat{\varepsilon} c_{21} + {} + \varepsilon \ln \hat{\varepsilon} \hat{c}_{22} - \varepsilon^2 c_{23} + \varepsilon^2 \hat{c}_{23}\,. \tag{34b}$$

The system of equations defined above has now a solution since its number of parameters is greater than the number of equations. The trick is obviously done by the second order bilogarithmic terms which become order unity in the overlap region.

In AFZAL's analysis, it was not clear that solutions of the form (29a) to (31b) could satisfy the matching conditions since the density profiles were omitted. Furthermore, in that analysis the bilogarithmic terms were obtained by assuming that the Reynolds stress terms could be expanded in a particular form, whereas here these bilogarithmic terms arise from the solution of the equations of motion.

As shown by the equations of motion of second order, the assumption that second-order bilogarithmic terms occur for compressible flow is reasonable and this is a direct consequence of the fact that the first order approximation solution contains

Table 1. Matching conditions

			zeroth order	first order		second order		
tangential velocity	outer exp ($y \to 0$)	$u =$	U_{10}	$+\varepsilon a_{11} \ln y$	$+\varepsilon a_{12}$	$+\varepsilon^2 a_{21} \ln^2 y$	$+\varepsilon^2 a_{22} \ln y$	$+\varepsilon^2 a_{23}$
	inner exp ($\hat{y} \to \infty$)	$\hat{u} =$	$+\varepsilon \ln \hat{\varepsilon} \hat{a}_{11}$ $+\varepsilon^2 \ln^2 \hat{\varepsilon} \hat{a}_{21}$	$+\varepsilon \hat{a}_{11} \ln y$ $+2\varepsilon^2 \ln \hat{\varepsilon} \hat{a}_{21} \ln y$	$+\varepsilon \hat{a}_{12}$ $+\varepsilon^2 \ln \hat{\varepsilon} \hat{a}_{22}$	$+\varepsilon^2 \hat{a}_{21} \ln^2 y$	$+\varepsilon^2 \hat{a}_{22} \ln y$	$+\varepsilon^2 \hat{a}_{23}$
temperature	outer exp ($y \to 0$)	$t =$	T_{10}	$+\varepsilon b_{11} \ln y$	$+\varepsilon b_{12}$	$+\varepsilon^2 b_{21} \ln^2 y$	$+\varepsilon^2 b_{22} \ln y$	$+\varepsilon^2 b_{23}$
	inner exp ($\hat{y} \to \infty$)	$\hat{t} =$	t_w $+ \varepsilon \ln \hat{\varepsilon} \hat{b}_{11}$ $+ \varepsilon^2 \ln^2 \hat{\varepsilon} \hat{b}_{21}$	$+\varepsilon \hat{b}_{11} \ln y$ $+2\varepsilon^2 \ln \hat{\varepsilon} \hat{b}_{20} \ln y$	$+\varepsilon \hat{b}_{12}$ $+\varepsilon^2 \ln \hat{\varepsilon} \hat{b}_{22}$	$+\varepsilon^2 \hat{b}_{21} \ln^2 y$	$+\varepsilon^2 \hat{b}_{22} \ln y$	$+\varepsilon^2 \hat{b}_{23}$
density	outer exp ($y \to 0$)	$\varrho =$	P_{10}	$-\varepsilon \frac{\mathrm{P}_{10}}{T_{10}} \hat{b}_{11} \ln y$	$-\varepsilon \frac{\mathrm{P}_{10}}{T_{10}} \hat{b}_{12}$	$+\varepsilon^2 \hat{c}_{21} \ln^2 y$	$+\varepsilon^2 \hat{c}_{22} \ln y$	$+\varepsilon^2 \hat{c}_{23}$
	inner exp ($\hat{y} \to \infty$)	$\hat{\varrho} =$	ϱ_w $+\varepsilon \ln \hat{\varepsilon} \frac{\varrho_w}{t_w} \hat{b}_{11}$ $+\varepsilon^2 \ln^2 \hat{\varepsilon} \hat{c}_{21}$	$-\varepsilon \frac{\varrho_w}{t_w} \hat{b}_{11} \ln y$ $+ 2\varepsilon^2 \ln \hat{\varepsilon} \hat{c}_{21} \ln y$	$-\varepsilon \frac{\varrho_w}{t_w} \hat{b}_{12}$ $+\varepsilon^2 \ln \hat{\varepsilon} \hat{c}_{22}$	$+\varepsilon^2 \hat{c}_{21} \ln^2 y$	$+\varepsilon^2 \hat{c}_{22} \ln y$	$+\varepsilon^2 \hat{c}_{23}$

logarithmic terms. Higher order asymptotic theories should then include such terms.

V. Conclusion

In this work we have shown that application to compressible turbulent boundary layers of the intermediate variable technique and of the matched asymptotic expansion method indicates that the defect and the wall layer solutions have an overlap domain.

The arguments of other authors based purely on considerations of the velocity profiles are not enough to settle the matching question. Indeed, if no further comments about the nature of the density expressions were necessary the following simple analysis would apply. In essence, what one has to show is that the difference between the outer and the inner approximations converges uniformly to zero in an overlap domain. In other words, one has to show that for each $\delta > 0$ there exists an $\varepsilon_\delta > 0$, such that if $\varepsilon < \varepsilon_\delta$ and $\mu(\varepsilon) \leqq x \leqq \nu(\varepsilon)$, $\mu(\varepsilon)$ and $\nu(\varepsilon)$ two class function defined in D_{overlap}, then $|w(x, \varepsilon)| = |u_{\text{inner}} - u_{\text{outer}}| < \delta$.

Now assuming that for the incompressible case there exists an overlap domain, it follows that

$$\forall \delta' > 0\,, \quad \exists\, \varepsilon'_\delta > 0\,,\ \varepsilon < \varepsilon'_\delta \wedge \mu(\varepsilon) \leqq x \leqq \nu(\varepsilon) \to \to |u_{i_{\text{inner}}} - u_{i_{\text{outer}}}| < \delta'\,.$$

This result and the fact that sine is a Lipschitz function, that is there exists a $K > 0$ (the constant of Lipschitz) such that, for each $x, y \in R$, we have $|\sin x - \sin y| < K|x - y|$ (for example, take $K = 1$), imply that if for the compressible case we take $\varepsilon_\delta = \varepsilon'_\delta$ and $\delta = \delta' K$, then for $\varepsilon < \varepsilon_\delta$ and $\mu(\varepsilon) \leqq x \leqq \nu(\varepsilon)$, μ and ν the same class of function above, we have

$$|u_{\text{inner}} - u_{\text{outer}}| = |\sin u_{i_{\text{inner}}} - \sin u_{i_{\text{outer}}}| < < K|u_{i_{\text{inner}}} - u_{i_{\text{outer}}}| < K\delta' < \varepsilon\,.$$

This would complete the proof that for the compressible case the inner and the outer expansion have an overlap domain which contains the overlap domain of the incompressible case. Indeed, this is also indicated by the analysis using the intermediate variable technique.

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A Variational Principle for an Isothermal Chemical Reaction

1. Introduction

Let A be a substance contained in a gas that flows through a cylindrical pore of a catalyst. On the walls of the pore the substance A transforms into another substance, say B. We assume that the transformation of A into B is an isothermal chemical reaction of the first order [1]. Then, after several changes of variables, the equation describing the dimensionless concentration of the substance A along the (dimensionless) length t of the catalyst pore can be written as [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{q}) + \lambda \frac{1 - q}{1 + \nu(1 - q)} = 0\,, \qquad t \in (0, 1)\,, \tag{1}$$

where $(.)^{\cdot} = \mathrm{d}/\mathrm{d}t\,(.)$, q is the dimensionless concentration, $a(t) > 0$ is the dimensionless diffusion coefficient and $\lambda \geqq 0$ and $\nu > 0$ are coefficients depending on temperature and initial concentration, respectively.

The boundary conditions corresponding to (1) are

$$q(0) = q(1) = 0\,. \tag{2}$$

The boundary value problem (1), (2) has been treated numerically in [2]. Also in [2] it is shown that (1), (2) has for any $\lambda \geqq 0$ a unique $C^2[0, 1]$ solution and that the following bounds hold [2 p. 53]:

$$\begin{aligned} &0 < q(t) < 1 \quad \text{for} \quad t \in (0, 1)\,, \qquad \lambda > 0\,, \\ &q(t) \equiv 0 \quad \text{for} \quad \lambda = 0\,. \end{aligned} \tag{3}$$

Our intention in this note is to treat the boundary value problem (1), (2) by a variational method developed in [3]. First, we shall show that the functional constructed by the procedure described in [3] attains a global minimum on the solution of (1), (2). Then, we shall use the Ritz method to determine approximate solutions to (1), (2) for several values of the parameters λ and ν. Finally, we shall present error estimates for the approximate solutions. The error estimates will be based on the values of the functional calculated on approximate solutions.

2. Variational principle and error estimate

To escort the procedure from [3] we first note that (1), (2) is equivalent to the stationarity condition ($\delta I_1 = 0$) for the functional I_1 given by

$$I_1 = \int_0^1 L\,\mathrm{d}t\,, \qquad L = \frac{a}{2}\dot{q}^2 - \frac{\lambda}{\nu} q - \frac{\lambda}{\nu^2} \ln\left[1 + \nu(1 - q)\right]. \tag{4}$$

By using L, given by $(4)_2$, we can follow the procedure of [3] to obtain the following functional

$$I = \int_0^1 \left\{ a\dot{Q}^2 - \frac{\lambda}{\nu} Q - \frac{\lambda}{\nu^2} \ln\left[1 + \nu(1 - Q)\right] + + \frac{\lambda}{\nu^2} \ln \frac{\lambda}{\lambda + \nu(\dot{a}\dot{Q} + a\ddot{Q})} + \frac{\lambda + (1 + \nu)(\dot{a}\dot{Q} + a\ddot{Q})}{\nu} \right\} \mathrm{d}t\,. \tag{5}$$