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Approximate Solutions to Singular Perturbation Problems: The Intermediate Variable Technique

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The intermediate variable technique, developed by S. Kaplun, "Fluid Mechanics and Singular Perturbations," Academic Press, San Diego, 1967, and by P. A. Lagerstrom and R. G. Casten, Basic concepts underlying singular perturbation techniques, *SIAM Rev.* **14** (1972), 63–120, for the solution of singular perturbation problems, is applied to several problems which are normally solved by other perturbative methods. The objective of the present analysis is to obtain approximate solutions which are characterized by their domains of validity, so that the matching of adjacent solutions is promptly determined. The analysis also shows how the intermediate equations can be derived and how they play an important role in the determination of approximate solutions.

1. INTRODUCTION

In many branches of mathematical physics, the governing equations are non-linear. Because of that, exact solutions are rare and so approximate analytical solutions to differential equations must be sought. The purpose of this work is to show how the intermediate variable technique [1, 2] can be used to obtain an approximation to the solution of a differential equation whose highest derivative is multiplied by a small parameter ε. This technique differs from other classical perturbative methods, such as the matched asymptotic expansion method and the strained coordinates method [1, 3], in the sense that it resorts to the concept of an intermediate limit to find the approximate equations, and that this process yields equations which are associated with different parts of the domain. Thus the intermediate variable technique avoids the conceptual difficulties of choosing the appropriate asymptotic expansions, and of determining the stretching functions. The emphasis of this technique is on characterizing the approximate equations by their domains of validity so that overlapping is promptly determined. However, the most important feature of this techni-

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que is its simplicity, which makes it a powerful means of obtaining approximations for engineering problems. In this paper, we show how apparently complicated problems, which are normally solved using other perturbative methods, are easily solved by the intermediate variable technique. These problems range from an exactly soluble differential equation to the complex problem of an incompressible turbulent flow past a flat surface. A novelty of the present analysis is the derivation of the intermediate equations which are otherwise omitted. The analysis shows that, under some conditions, these equations can be used to find good approximate solutions. To the present authors' knowledge, this is the first time that such a feature of the intermediate variable technique is being explored. The technique, as proposed here, is not to be seen as a substitute method to other classical perturbation methods, but as a technique which can be of great help in the pre-analysis of singular perturbation problems.

2. The Intermediate Variable Technique

The objective of the intermediate variable technique is to split a differential equation into a set of approximate equations which are uniformly valid in different parts of the domain. The approximate equations are obtained by applying an arbitrary limit process to the differential equation of the problem. Normally, the splitting of the equations gives first-order approximations only. However, an extension of the splitting may be obtained for higher orders by the introduction of a fictitious perturbation of an arbitrary order ε, Kaplun [1]. This procedure makes the use of arbitrary limit processes much more complicated and difficult to understand. Because of that, and despite realizing that such an extension provides a great deal of additional information about the domains of validity, we have opted in this work to not consider the splitting of a differential equation for higher orders.

The η-limit of an equation $E(x, y; \varepsilon)$ is defined as follows. Let the intermediate variable, $\hat{x}$, be

$$\hat{x}\eta(\varepsilon) = x, \tag{1}$$

where, as indicated, η is an arbitrary function of ε. Then, the η-limit of $E(x, y; \varepsilon)$ is

$$\lim_\eta E(x, y; \varepsilon) = \lim E(\hat{x}\eta(\varepsilon), y; \varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \text{ with } \hat{x} \text{ fixed.} \tag{2}$$

The above definition is of fundamental importance for what follows and so it should be well understood. It will be studied in more detail, together with some important concepts, in the next examples.

EXAMPLE 1. Consider the exactly soluble boundary layer problem given by

$$\varepsilon y'' + (1+\varepsilon)\, y' + y = 0, \qquad y(0)=0,\ y(1)=1 \tag{3}$$

To find an approximation to Eq. (3) we substitute expression (1) above and obtain

$$\frac{\varepsilon}{\eta^2}\, \hat{y}'' + \frac{(1+\varepsilon)}{\eta}\, \hat{y}' + \hat{y} = 0, \tag{4}$$

where $\hat{y}$ and $\hat{y}'$ denote $y(\eta\hat{x})$ and $d\hat{y}/d\hat{x}$, respectively.

Depending on the order of η, each term will have a formal order in ε. The derivatives and $\hat{x}$ are considered to be formally of order unity. For example, the first term is formally of order $\varepsilon\eta^{-2}$.

Applying the limit process η-limit to Eq. (4), we find the following approximate equations for the various orders of η:

$$O(\eta) = 1, \qquad \hat{y}' + \hat{y} = 0 \tag{5a}$$

$$O(\varepsilon) < O(\eta) < 1, \qquad \hat{y}' = 0 \tag{5b}$$

$$O(\eta) = O(\varepsilon), \qquad \hat{y}'' + \hat{y}' = 0 \tag{5c}$$

$$O(\eta) < O(\varepsilon), \qquad \hat{y}'' = 0. \tag{5d}$$

Equations (5a) and (5c) exhibit an important feature, that is, if η is such that the limit process η-limit applied to Eq. (4) gives Eq. (5b), then the η-limit of Eq. (5a) and of Eq. (5c) also give Eq. (5b). These two equations are then said to be "rich enough" to contain Eq. (5b). Lagerstrom and Casten [2] make the following definitions:

DEFINITION I. If E is an equation and $\lim_{\eta_1} \mathrm{E} = \mathrm{E}_1$, $\lim_{\eta_2} \mathrm{E} = \mathrm{E}_2$ and also $\lim_{\eta_2} \mathrm{E}_1 = \mathrm{E}_2$, we say that E_1 contains E_2 (relative to E).

DEFINITION II. The *formal domain* of validity of an equation F, relative to the "full" equation E, is the ord η such that $\lim_{\eta} \mathrm{E}$ is either F or an equation contained in F.

Thus, according to these definitions, the domain of validity of Eqs. (5a) and (5c) are respectively

$$D_0 = \{\eta / \mathrm{ord}\ \eta > \mathrm{ord}\ \varepsilon\} \tag{6a}$$

and

$$D_i = \{\eta / \mathrm{ord}\ \eta < \mathrm{ord}\ 1\}. \tag{6b}$$

Equations (5a) and (5c) are the two important equations; they are, in fact, called the principal equations. They define the outer, (5a), and the inner, (5c), equations, and the thickness of the boundary layer (ε). We shall call the other equations, (5b) and (5d), the intermediate equations.

Now, if some sort of relationship can be established between the formal domains defined by sets (6a–6b) and the actual domains of validity (as an approximation to the solution of Eq. (4)) of the solutions of Eqs. (5a) and (5c), one can easily justify the matching of the outer and the inner solutions. Using the notion that neighboring equations should yield neighboring solutions, Kaplun [1] enunciated the following heuristic principle:

PRINCIPLE. *If y is a solution of an equation* E *and* E* *is an approximate equation, then there exists a solution y* of* E* *whose actual domain of validity (as an approximation to y) includes the* formal domain *of validity of* E* *(as an approximation to* E).

The above principle is not always true and, in particular, it fails when small terms have large integrated effects [2]. In the present case, however, the principle can be applied. The overlap domain is then given by

$$D_0 \cap D_i = \{\eta/\varepsilon \ll \eta \ll 1\}. \tag{7}$$

We now finally proceed to the solution of problem (3). From Eqs. (5a) and (5c), it follows that the outer and the inner solutions are respectively

$$f(\hat{x}) = Ae^{-\hat{x}} \tag{8a}$$

and

$$g(\hat{x}) = B + Ce^{-\hat{x}}. \tag{8b}$$

Since domain (6a) includes the point $x = 1$, Eq. (8a) must satisfy the boundary condition $y(1) = 1$. This gives

$$f(x) = e^{1-x}. \tag{9a}$$

The inner solution in its turn must satisfy the boundary condition at $x = 0$, this gives

$$B + C = 0. \tag{10}$$

We still need a complementary condition to find B and C. We know that the outer and the inner solutions must match in the overlap domain (7). Thus, we must have

$$\lim_{\eta} f(x) = \lim_{\eta} g(x/\varepsilon), \qquad \varepsilon \ll \eta \ll 1. \tag{11}$$

Note that in writing (11) we have used $\hat{x} = x$ for the outer solution (Eq. (5a)) and $\hat{x} = x/\varepsilon$ for the inner solution (Eq. (5c)). From expressions (10) and (11) it follows that

$$B = e, \qquad C = -e. \tag{12}$$

So far, we have found *two* expressions which approximate the solution of the problem in the entire region. So, in theory, we have a solution which is uniformly valid in [0, 1]. However, it is often of practical interest to represent the solution by *one* expression, *h*. To construct this expression, the so-called composite expression, we use here the additive composite method. In this method, the outer and the inner expressions are added and the part they have in common is subtracted, so that it is not evaluated twice. In the overlap domain Eq. (9a) assumes the value

$$\lim f(x) = e, \qquad x \to 0. \tag{13}$$

Hence, it follows that

$$h(x) = e^{1-x} - e^{1-x/\varepsilon}. \tag{14}$$

As just illustrated, the existence of an overlap domain is crucial to the matching of the outer and of the inner solutions. Unfortunately, the calculation of this domain is often too difficult, and so, as long as there is no inconsistency in the performance of the matching of the outer and inner solutions, one is normally happy in just assuming that there is such a region.

In the next section we show how the search for the overlap region normally yields a large number of qualitative information about the problem under consideration. The analysis also shows that the intermediate equations may be used sometimes to great advantage.

3. Results

To demonstrate the versatility of the intermediate variable technique, consider the problems:

Example 2. Problem with a boundary layer of thickness $\varepsilon^{1/2}$.

$$\varepsilon y'' - x^2 y' - y = 0, \qquad y(0) = y(1) = 1. \tag{15}$$

Applying the limit process η-limit to Eq. (15) we obtain

$$O(\eta)=1, \qquad -\hat{x}^2\hat{y}'-\hat{y}=0 \tag{16a}$$

$$O(\varepsilon^{1/2})<O(\eta)<1, \qquad \hat{y}=0 \tag{16b}$$

$$O(\eta)=O(\varepsilon^{1/2}), \qquad \hat{y}''-\hat{y}'=0 \tag{16c}$$

$$O(\eta)<O(\varepsilon^{1/2}), \qquad \hat{y}''=0. \tag{16d}$$

The two principal equations, easily spotted, are Eqs. (16a) and (16c). Consequently the outer and the inner variables are $\hat{x}=x$ and $\hat{x}=x/\varepsilon^{1/2}$, respectively. Note that for this case the thickness of the boundary layer is $\varepsilon^{1/2}$.

The general solution to Eq. (16a) is

$$\hat{y}=Ae^{1/\hat{x}}, \tag{17a}$$

where, to satisfy the boundary condition $y(1)=1$, we have $A=e^{-1}$.

The inner solution (Eq. (16c)) in its turn is given by

$$\hat{y}=Be^{\hat{x}}+Ce^{-\hat{x}}, \tag{17b}$$

where, the boundary condition $y(0)=1$ implies that $B+C=1$.

The two parameters B and C must be determined as in the previous section, that is, by matching solutions (17a)–(17b). Here, we start to suspect that something must have gone terribly wrong with our analysis, since the outer solution blows up as $\hat{x}$ tends to zero and hence no matching can be performed. Indeed, we have overlooked the fact that the function multiplying the first derivative in Eq. (15), x^2, becomes zero at $\hat{x}=0$. This makes the outer solution singular at $\hat{x}=0$ except when $\hat{y}(0)=0$, that is, $A=0$. However, we have seen that A must be e^{-1} so that we have $y(1)=1$ satisfied. What in fact happens here is that we have a boundary layer of thickness ε at $x=1$, whose solution should be able to satisfy $y(1)=1$ and match with solution (17a). The existence of the right boundary layer is indeed suggested by Eq. (16b). This equation states that the approximate solution, $\hat{y}=0$, applies over a large portion of the domain, $\varepsilon^{1/2}\ll\eta\ll 1$, so that the actual solution must "jump" near 1 to satisfy $y(1)=1$. As a matter of fact the two intermediate equations yield

$$\hat{y}=0 \qquad \text{in} \quad D_0=\{\eta/\varepsilon^{1/2}\ll\eta\ll 1\} \tag{18a}$$

and

$$\hat{y}=ax+b \qquad \text{in} \quad D_i=\{\eta/\eta\ll\varepsilon^{1/2}\}. \tag{18b}$$

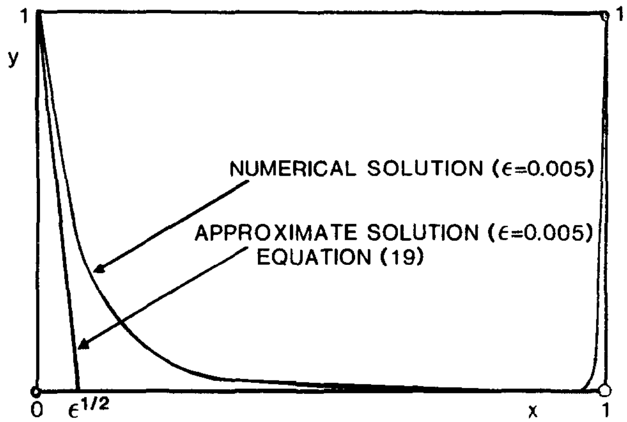

FIG. 1. Comparison of exact and approximate results.

This is an important result which gives us an approximate solution for the whole of the domain except in the neighbourhoods of $x=1$ and of $x=\varepsilon^{1/2}$. To get an approximation for the interval $[0, 1]$ we extend solutions (18a)–(18b) to the points $x=1$, $x=\varepsilon^{1/2}$, obtaining

$$h=\begin{cases}\varepsilon^{-1/2}x+1, & x\leqslant\varepsilon^{1/2}\\ 0, & \varepsilon^{1/2}<x<1\\ 1, & x=1.\end{cases} \tag{19}$$

Figure 1 compares the above approximation with the actual solution of Eq. (15) for $\varepsilon=0.005$. The agreement is reasonable. Of course, the agreement becomes better as ε tends to zero.

EXAMPLE 3. Boundary layer with a singularity.

$$\varepsilon y''+\frac{y'}{x}+y=0, \qquad \begin{array}{l} y(1)=e^{-1/2}\\ y(0)=0.\end{array} \tag{20}$$

To solve this problem we apply the η-limit to Eq. (20) to obtain

$$O(\eta)=1, \qquad \hat{y}'/\hat{x}+\hat{y}=0 \tag{21a}$$

$$O(\eta)<1, \qquad \hat{y}'/\hat{x}=0. \tag{21b}$$

Observe that no second-order approximate differential equation occurs no matter what is the order of $\eta(\varepsilon)$. Therefore, a complete approximate solution to problem (20) seems to be impossible to obtain through the present procedure. In fact, this difficulty is inherent to the problem and occurs with any perturbation method used. The singularity at $x=0$ of the term $1/x$ multiplying the first derivative in Eq. (20) is, of course, the cause of our difficulties. To avoid this singularity we redefine the intermediate variable as

$$\hat{x}=(x-1)/\eta(\varepsilon). \tag{22}$$

Substitution of the above equation into Eq. (20) gives

$$\frac{\varepsilon}{\eta}\,\hat{y}''+\frac{\hat{y}'}{1+\hat{x}\eta}+\eta\hat{y}=0. \tag{23}$$

Now, passing the limit process η-limit in expression (23) we obtain

$$O(\eta)=1, \qquad \hat{y}'+(1+\hat{x})\,\hat{y}=0 \tag{24a}$$

$$O(\varepsilon)<O(\eta)<1, \qquad \hat{y}'=0 \tag{24b}$$

$$O(\eta)=O(\varepsilon), \qquad \hat{y}''+\hat{y}'=0 \tag{24c}$$

$$O(\eta)<O(\varepsilon), \qquad \hat{y}''=0. \tag{24d}$$

The two principal equations are obviously given by Eq. (24a), the outer, and Eq. (24c), the inner. The outer equation subject to the boundary condition $y(1)=e^{-1/2}$, gives

$$\hat{y}=e^{-(1+\hat{x})^2/2}, \tag{25a}$$

whereas, the inner equation subject to the other boundary condition gives

$$\hat{y}=Ae^{-\hat{x}}+B, \tag{25b}$$

$$Ae^{1/\varepsilon}+B=0. \tag{25c}$$

Comparison of Eqs. (25a) and (25b) in the overlap region, $\varepsilon \ll \eta \ll 1$, determines $B=1$, so that $A=-e^{-1/\varepsilon}$. The composite solution constructed using Eqs. (25a) to (25c) is then given by

$$\hat{y}=-e^{-x/\varepsilon}+e^{-x^2/2}. \tag{26}$$

This example illustrates well the capability of the intermediate variable technique of anticipating difficulties in the search for the solutions of perturbation problems. Thus the necessity of using some sort of co-ordinate transformation is evident from Eqs. (21a)–(21b). Since the intermediate equations are arrived at with great ease, their derivation is worthy and avoids unnecessary work.

EXAMPLE 4. Nested boundary layers. Consider the problem defined by

$$\varepsilon^3 xy'' + x^2y' - y(x^3+\varepsilon) = 0, \qquad y(0)=1,\ y(1)=\sqrt{\varepsilon}. \tag{27}$$

Substitution of the intermediate variable, $x = \hat{x}\eta(\varepsilon)$, gives

$$\frac{\varepsilon^3}{\eta}\hat{x}\hat{y}'' + \eta\hat{x}^2\hat{y}' - \hat{y}(\eta^3\hat{x}^3+\varepsilon) = 0. \tag{28}$$

Passing the η-limit in Eq. (28) we obtain

$$O(\eta)=1, \qquad \hat{x}^2\hat{y}' - \hat{y}\hat{x}^3 = 0 \tag{29a}$$

$$O(\varepsilon) < O(\eta) < 1, \qquad \hat{x}^2\hat{y}' = 0 \tag{29b}$$

$$O(\eta) = O(\varepsilon), \qquad \hat{x}^2\hat{y}' - \hat{y} = 0 \tag{29c}$$

$$O(\varepsilon^2) < O(\eta) < O(\varepsilon), \qquad \hat{y} = 0 \tag{29d}$$

$$O(\eta) = O(\varepsilon^2), \qquad \hat{x}\hat{y}'' - \hat{y} = 0 \tag{29e}$$

$$O(\eta) < O(\varepsilon^2), \qquad \hat{x}\hat{y}'' = 0. \tag{29f}$$

The particularity of this example is the occurrence of *three* principal equations, Eqs. (29a), (29c), and (29e). This implies that now we have two boundary layers, one of thickness ε and another of thickness ε^2, both of them at $x=0$. The necessity of working with a set of *two* inner equations is not clear at all from Eq. (27), and any attempt by a inexperienced user of perturbation methods to apply a classical method, such as the matched asymptotic expansion method, to Eq. (27) would probably lead to an unsuccessful first-try. Our straightforward analysis, however, quickly reveals the structure of the domains of the several approximations. Thus, solutions of Eqs. (29a) and (29c) must match in the overlap region, $\varepsilon \ll \eta \ll 1$, whereas solutions of Eqs. (29c) and (29e) must match in the overlap region, $\varepsilon^2 \ll \eta \ll \varepsilon$. The general solution of (29a) with the boundary condition $y(1) = \sqrt{\varepsilon}$ gives

$$\hat{y} = e^{-\hat{x}^2/2}. \tag{30a}$$

The outermore inner solution (Eq. (29c)) is given by

$$\hat{y} = Ae^{-1/\hat{x}}. \tag{30b}$$

Comparison of Eq. (30a) as $\hat{x}$ tends to zero with Eq. (30b) as $\hat{x}$ tends to infinity determines $A=1$. It is obvious that Eq. (30b) does not satisfy the boundary condition at $x=0$. This is done by Eq. (29e) whose solution is

$$\hat{y} = B\hat{x}^{1/2}I_1(2\hat{x}^{1/2}) + C\hat{x}^{1/2}K_1(2\hat{x}^{1/2}), \tag{30c}$$

where I_1 and K_1 are solutions of the modified Bessel equation.

The matching of Eqs. (30b) and (30c) yields

$$B=0, \qquad C=2.$$

Hence, the approximate solution of Eq. (27) is

$$\hat{y}=e^{-x/2}+e^{-\varepsilon/x}-1+2x^{1/2}\varepsilon^{-1}K_1(2x^{1/2}\varepsilon^{-1}). \tag{31}$$

EXAMPLE 5. Turbulent boundary layer. Consider now the problem of a two-dimensional, incompressible, turbulent fluid flowing over a flat surface. The equations of motion for a steady flow can be written as

(a) continuity

$$\partial\bar{u}_i/\partial x_i=0 \tag{32a}$$

(b) momentum

$$\frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_i}=\frac{-1}{\rho}\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(-\varepsilon^2\,\overline{u_i'u_j'}+\varepsilon^2\hat{\varepsilon}\frac{\partial\bar{u}_i}{\partial x_j}\right). \tag{32b}$$

In these equations, x_i, $i=1,2$, are the tangential and normal coordinates to the surface, u_i are the velocities, p is the pressure, and ρ is the density. The superscript denotes turbulent fluctuations and the bars denote conventional time averaging. A summation is understood for repeated indices. All quantities are non-dimensional based on properties evaluated at external condition.

The small parameter, ε, is introduced assuming [4] that the fluctuations are of the order of the non-dimensional friction velocity, u_τ, that is

$$\varepsilon=O(u_i')=O(u_\tau/u_\infty)=O[(1/u_\infty)\sqrt{\tau_\omega/\rho_\omega}]. \tag{33}$$

Here, u_∞ and L are reference quantities and τ_ω is the total shear stress at the wall. The second small parameter, $\hat{\varepsilon}$, arises from the fact that the kinematic viscosity, ν, is small and so viscous terms are only important close to the wall. Hence we have

$$\varepsilon^2\hat{\varepsilon}=\frac{\nu}{u_\infty L}=\frac{1}{R}, \tag{34}$$

where R denotes the Reynolds number.

The solution of Eqs. (32a)–(32b) is required to satisfy the no-slip and no-permeability conditions at the wall, that is,

$$\bar{u}_1(x_1,0)=\bar{u}_2(x_2,0)=0. \tag{35}$$

Upstream the flow has to agree with some previously prescribed velocity profile.

Substitution of the intermediate variable

$$\hat{x}_2\eta(\varepsilon) = x_2 \tag{36}$$

into Eq. (32b) together with the fact that Eq. (32a) gives $\partial\bar{u}_2/\partial x_2 = O(\partial\bar{u}_1/\partial x_1)$, yields for the various orders of η:

(a) momentum equation in tangential direction:

$$O(\eta) = 1, \qquad \bar{u}_1\frac{\partial\bar{u}_1}{\partial x_1} + \bar{u}_2\frac{\partial\bar{u}_1}{\partial\hat{x}_2} = \frac{-1}{\rho}\frac{\partial\bar{p}_1}{\partial x_1} \tag{37a}$$

$$O(\varepsilon^2) < O(\eta) < 1, \qquad \bar{u}_1\frac{\partial\bar{u}_1}{\partial x_1} + \bar{u}_2\frac{\partial\bar{u}_1}{\partial\hat{x}_2} = \frac{-1}{\rho}\frac{\partial\bar{p}_1}{\partial x_1} \tag{37b}$$

$$O(\eta) = O(\varepsilon^2), \qquad \bar{u}_1\frac{\partial\bar{u}_1}{\partial x_1} + \bar{u}_2\frac{\partial\bar{u}_1}{\partial\hat{x}_2} = \frac{-1}{\rho}\frac{\partial\bar{p}_1}{\partial x_1} + \frac{\partial\tau_{xy}}{\partial\hat{x}_2} \tag{37c}$$

$$O(\hat{\varepsilon}) < O(\eta) < O(\varepsilon^2), \qquad \frac{\partial\tau_{xy}}{\partial\hat{x}_2} = 0 \tag{37d}$$

$$O(\eta) = O(\hat{\varepsilon}), \qquad \frac{\partial\tau_{xy}}{\partial\hat{x}_2} + \frac{\partial^2\bar{u}_1}{\partial\hat{x}_2^2} = 0 \tag{37e}$$

$$O(\hat{\varepsilon}) < O(\eta), \qquad \frac{\partial^2\bar{u}_1}{\partial\hat{x}_2^2} = 0, \tag{37f}$$

where τ_{xy} denotes $\overline{-u_1' u_2'}$.

(b) momentum equation in normal direction:

$$O(\eta) = 1, \qquad \partial\bar{p}_2/\partial\hat{x}_2 = 0 \tag{38a}$$

$$O(\eta) < 1, \qquad \partial\bar{p}_2/\partial\hat{x}_2 = 0. \tag{38b}$$

The principal equations are Eqs. (37a), (37c), and (37e). They show that far away from the wall, in the outermost layer, the Reynolds stress and the viscous stress effects can be neglected. Approaching the wall, a layer is reached where the inertia, pressure and Reynolds stress effects are of the same importance. Further down, very close to the wall, a new region is reached where the Reynolds stress and the viscous stress effects are the dominating effects. Thus, the approximate solution to this problem consists of three solutions which must be matched in the overlap domains defined by Eqs. (37b) and (37d). The x_2-momentum equation just shows that to the lowest order, pressure is constant across the boundary layer.

The difficulty with this example is that the principal equations are still complicated equations. To solve them we first need to introduce a turbulence model to relate the Reynolds stress to the mean velocity field. For the sake of simplicity we use the mixing-length theory of Prandtl together with the Van Driest assumption about the form of the mixing-length, l. Hence it follows that

$$\tau_{xy} = l^2 \left[\frac{\partial \bar{u}_1}{\partial x_2} \right]^2 \tag{39a}$$

with

$$l = K\hat{x}_2[1 - \exp(-\hat{x}_2/A)], \qquad A = \text{constant}. \tag{39b}$$

Substitution of Eqs. (39a)–(39b) into Eq. (37e), subject to the boundary condition $\bar{u}_1(x_1, 0) = 0$ and the assumption that close to the wall the total shear stress is nearly constant, yields

$$\overline{u_1} = \int_0^{\hat{x}_2} \frac{2\varepsilon}{1 + \sqrt{1 + 4B}}\, d\hat{x}_2, \tag{40}$$

where $B = l^2\varepsilon^2$.

Equation (40) gives a continuous velocity distribution in the domain $\eta \ll \varepsilon^2$ defined by Eq. (37e). Very close to the wall, in the so-called wall layer, Eq. (40) reduces to

$$\bar{u}_1 = \varepsilon\hat{x}_2 \tag{41}$$

as implied by Eq. (37f).

In the region of fully turbulent flow, $\hat{\varepsilon} \ll \eta \ll \varepsilon^2$, it reduces to

$$\bar{u}_1 = \varepsilon(K^{-1} \ln \hat{x}_2 + C). \tag{42}$$

The above equation is the famous law of the wall. It could have been derived directly from the *intermediate* Eq. (37d). Note that classical perturbation methods would have omitted the two important intermediate Eqs. (37d) and (37f).

A complete solution for the entire flow region depends now on the solution of Eqs. (37a) and (37c). These equations are complex and it is not our aim to solve them here. Indeed, in this example we have striven to show how the intermediate variable technique can be used to obtain valuable qualitative information about the inner layers that arise in a turbulent boundary layer problem.

4. Final Remarks

In this work, we have not tried to propose a substitutive method to classical perturbation methods, but have presented a method which can be used with great advantage in a pre-analysis of singular perturbation problems. The method, as introduced here, is not suitable for finding higher order approximations, but provides very good lower order results. The emphasis is on obtaining qualitative results and on characterizing approximate solutions by their domains of validity. In particular, we have shown how the intermediate equations can sometimes provide useful approximations. Another interesting feature of this method is the possibility of coupling it with numerical techniques. Thus the principal equations of Example 5 could be solved numerically with a patching of adjacent solutions being performed in the overlap regions.

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