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ON KAPLUN LIMITS AND THE MULTILAYERED ASYMPTOTIC STRUCTURE OF THE TURBULENT BOUNDARY LAYER

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In the present work, some formal properties of singular perturbation equations are studied through the concept of "equivalent in the limit" of Kaplun, so that a proposition for the principal equations is derived. The proposition shows that if there is a principal equation at a point (η, 1) of the ($\Xi \times \Sigma$) product space, Ξ space of all positive continuous functions in (0, 1], Σ = (0, 1], then there is also a principal equation at a point (η, ε) of ($\Xi \times \Sigma$), ε = first critical order. The converse is also true. The proposition is of great implication because it ensures that the asymptotic structure of a singular perturbation problem can be determined by a first-order analysis of the formal domains of validity. The turbulent boundary layer asymptotic structure is then studied by application of Kaplun limits to three test cases: the zero-pressure boundary layer, the separating boundary layer and the shock-wave interacting boundary layer. As it turns out, different asymptotic structures are found, depending on the test cases considered. However, before we consider the real turbulent boundary layer problem, the basics of the theory are illustrated by the study of a model equation that mimics turbulent flow passed over a flat surface. The model equation was chosen for being relatively simple while retaining most of the features of the real problem. This allows one to easily grasp the main concepts and ideas wthout being hampered by unnecessary details. Results show that a two-layered structure is derived, which, however, is different from the classic structure commonly found in the literature, and hence is capable of explaining the flow separation phenomenon. A skin-friction equation resulting from a matching process, and universal laws resulting from local approximated equations are carefully interpreted and evaluated.

INTRODUCTION

In physics and mathematics many phenomena are modeled through intricate equations that present no analytical solution. As such, engineers, physicists, and applied mathematicians are forced to develop techniques that yield approximated solutions to the problems with which they are faced. These techniques often end in sophisticated procedures that in only a very few cases are restrained to fully analytical frameworks. In most cases, analytical procedures have to be combined with numerical procedures to produce an approximated solution.

Perturbation methods have evolved along the last 40 years into a powerful tool for solving a large class of complex problems. They have, therefore, become a basic working tool of many engineers and applied mathematicians. In fact, a large number of papers can be found in the literature that use perturbation methods as their primary solution procedure.

The purpose of this work is twofold: (1) to consider more thoroughly some fundamental concepts and ideas used in solving perturbation problems, and (2) to study the turbulent boundary layer asymptotic structure by applying Kaplun limits to the Navier-Stokes equations.

While some precise definitions can be enunciated, and exact results obtained to find uniform approximations and to perform the matching of functions, the determination of the domain of validity of an approximation is always difficult. Two important results in perturbation theory are the intermediate matching lemma and the extension theorem of Kaplun. These results are of fundamental importance for the construction of matched asymptotic expansions, but say nothing about the domain of validity of the approximations. To circumvent this difficulty, Kaplun (1967) applied the concept of limit-processes directly to the equations rather than to the solutions and enunciated an Ansatz, the Ansatz about domains of validity, which relates the domain of validity of solutions with the formal domain of validity of equations (a concept that is easily defined). Examples are known where Kaplun's ideas fail; however, for some difficult problems, for example, the Stokes paradox of fluid mechanics, only consideration of these ideas can clarify the conceptual structure of the problem. Here, we study some formal properties of equations yielded by the definition of "equivalent in the limit" of Kaplun and relate them to the actual problems of determining the overlap domain and of matching asymptotic expansions. The concept of "richer than" of Kaplun and Lagerstrom (1957) is given a more elaborate interpretation that leads to the derivation of a theorem for the principal equations. The theorem shows that if there is a principal equation at a point $(\eta, 1)$ of the $(\Xi \times \Sigma)$ product space, Σ = space of all positive continuous functions on $(0, 1]$, $\Sigma = (0, 1]$, then there is also a principal equation at a point (η, ε) of $(\Xi \times \Sigma)$, ε = first critical order. The converse is also true. The consequence of this theorem is that no matter to what order of magnitude we want an approximation to be accurate, it is always possible to find high-order solutions at points $(\eta, 1)$ of the $(\Xi \times \Sigma)$ space (η = point of the Ξ space obtained through the passage of Kaplun's limit process, where a principal equation is located) that satisfy the required degree of accuracy, overlap, and cover the entire domain.

Because the basis of perturbation methods comes from heuristic ideas and intuitive concepts rather than some general theory, basic questions are normally answered by physical arguments and an acquired experience. Texts with an applied-oriented nature, such as the books of Cole (1968), Nayfeh (1973), Van Dyke (1975), and Kevorkian and Cole (1981), show through examples how these heuristic ideas can be put into practice and how some simple rules can be devised for the analysis of perturbation problems. These rules have become very popular over the years, but are known to fail in many situations. To formalize such rules and show the conditions under which they work, Eckhaus (1969, 1972, 1973) and Fraenkel (1969) carried out a more mathematically oriented analysis that had to be necessarily of a more limited scope. Further important works on perturbation methods with mathematically oriented approaches are the books by O'Malley (1974) and Wasow (1976), in which some exact results are derived for some types of ordinary differential equations.

In an attempt to make Kaplun's ideas clearer, Lagerstrom and Casten (1972) published a work in which a survey of some ideas on perturbation methods was presented. Again a heuristic approach was used. The work, however, presented some new definitions and results that were known to work for leading-order approximate solutions. Some of these results have been revisited recently in publications by Lagerstrom (1988) and by Silva Freire and Hirata (1990).

Most of Kaplun's ideas have been developed in the literature in connection with boundary value problems. Nipp (1988), however, used the concepts presented in Kaplun

(1967) and Lagerstrom and Casten (1972) to work out a systematic approach to solve a large class of singularly perturbed initial value problems. His analysis considers both the formal and the rigorous aspects of the problem, yielding a procedure to find formal approximations of order unity. Although many of Nipp's ideas carry over to boundary value problems, much work still needs to be done concerning the determination of higher-order approximate solutions for these problems.

The formal properties of equations studied here are aimed at boundary layer problems. The theorem of the principal equations formalizes the notion of a distinguished limit used so often in the literature, allowing Kaplun's ideas to be used in a systematic manner. The matched asymptotic expansions method, for example, depends on two crucial guesses for the determination of approximate solutions: the choice of the stretching function and the choice of the asymptotic expansions. These choices are normally guided by physical arguments, but are in the end always made by trial and error. In fact, the determination of the stretching function and of the asymptotic expansions has always been seen as an art. With the theorem of the principal equations, the stretching function is immediately found, whereas the appropriate gauge functions for the asymptotic expansions are obtained from Kaplun's concept of critical orders.

The asymptotic structure of the turbulent boundary layer has been investigated extensively by a number of authors in the last 20 years. Unlikely, in the laminar flow case, whose solution has been known since the 1960s, the turbulent problem poses some questions that still have to be understood and answered. Of course, all difficulties stem from the introduction of the time-averaged equations. To make these equations a determined system, closure conditions must be introduced to relate the Reynolds stresses to the mean flow velocities. The Reynolds stresses, the time averages of the fluctuating velocities, describe the effect of turbulent fluctuations on the mean flow; if they could be determined, the mean flow equations could be solved and the asymptotic structure unveiled. Many closure conditions have been proposed in literature, but, unfortunately, none of them are generally valid.

Using only the hypothesis that the order of magnitude of the Reynolds stresses does not change throughout the boundary layer, some authors (Yajnik [1970], Mellor [1972]) have found the turbulent boundary layer to have a two-deck structure consisting of a wall region and a defect region. Other authors using closure conditions in terms of eddy-viscosity (Bush and Fendell [1972]) or $\kappa - \varepsilon$ (Deriat and Guiraut [1986]) models have reached the same conclusion, making the two-deck asymptotic structure of the turbulent boundary layer the basis of most subsequent work.

Recently, however, there has been a claim that the turbulent boundary has instead a three-layered structure (Long and Chen [1981], Sychev and Sychev [1987], Melnik [1989]), and that this is the only structure that can possibly handle flows subject to pressure gradients.

In this work, the asymptotic structure of the turbulent boundary layer is investigated in two levels. First, a model equation is constructed that is expected to comply with many of the features of the physical problem. The model equation is a simplified realization of the boundary layer equations, with an algebraic turbulence model of the mixing length type. As we shall see, the application of Kaplun limits to this equation results in an approximate solution that results in a flow structure that closely resembles the real problem. Second, Kaplun limits will be applied directly to the Navier-Stokes equation. Three cases are investigated here: (1) turbulent flow over a flat surface with zero-pressure gradient, (2) separating turbulent boundary layer flow, and (3) the interaction of a normal shock wave with a turbulent boundary layer. In all three cases, the differences arising in flow structure are discussed in detail.

THE FUNDAMENTALS OF THE THEORY

We shall consider perturbation methods to find approximate solutions to differential equations of the form

$$\varepsilon\, E_1\left(x, y, \ldots, y^{(n)}\right) + E_2\left(x, y, \ldots, y^{(n-1)}, \varepsilon\right) + \ldots = 0 \tag{1}$$

that is, equations where the small parameter ε multiplies the highest derivative term. E_i is a given function of the variables $x, y, \ldots, y^{(n)}, \varepsilon$. Here $y^{(n)}$ is used to denote $d^n y/dx^n$.

The method to be studied here aims at developing a procedure to find approximate solutions to equations with form of Eq. (1) that are valid in different parts of the domain. This can be achieved by the introduction of a limit process that determines the terms of Eq. (1) that have a dominant effect in the various regions of the domain.

To define this limit process some basic concepts need to be introduced. The following topology is introduced on the collection of order classes (Meyer, 1967).

For positive, continuous functions of a single variable ε defined on (0, 1], let ord η denote the class of equivalence.

$$\text{ord}\,\eta = \{\theta(\varepsilon) \text{ such that } \lim \theta(\varepsilon)/\eta(\varepsilon), \quad \varepsilon \to 0, \quad \text{exists and is} \neq 0\} \tag{2}$$

A partial ordering is constructed on these functions by defining

$$\text{ord}\,\eta_1 < \text{ord}\,\eta_2 \iff \lim \frac{\eta_1}{\eta_2} = 0, \ \varepsilon \to 0 \tag{3}$$

A set D of order classes is said to be convex if ord δ_1, ord $\delta_2 \in D$ and ord $\delta_1 <$ ord $\theta <$ ord δ_2 together imply ord $\theta \in D$. A set D is said to be open if it is convex and if ord $\theta \in D$ implies the existence of functions γ, δ, such that ord $\theta >$ ord $\gamma \in D$ and ord $\theta <$ ord $\delta \in D$. A set D, on the other hand, is said to be closed if it is convex and has particular elements ord δ_1,ord δ_2 such that ord $\delta_1 \leq$ ord $\theta <$ ord δ_2 for every ord $\theta \in D$. Two order sets, D and D', are said to be adjacent if: (1) $D' > D$ and (2) $\eta < D'$ and $\eta' > D \;\to \eta' > \eta$. We may refer to D' as being the upper adjacent region of D. Analogously, D is said to be the lower adjacent region of D'.

Definition (Lagerstrom, 1988). We say that $f(x, \varepsilon)$ is an approximation to $g(x, \varepsilon)$ uniformly valid to order $\delta(\varepsilon)$ in a convex set D (f is a δ-approximation to g), if

$$\lim \frac{f(x, y) - g(x, y)}{\delta(\varepsilon)} = 0, \ \varepsilon \to 0, \text{ uniformly for } x \text{ in } D \tag{4}$$

The function $\delta(\varepsilon)$ is called a gauge function.

The essential idea of η-limit process is to study the limit as $\varepsilon \to 0$ not for fixed x near the singularity point x_d, but for x tending to x_d in a definite relationship to ε specified by a stretching function $\eta(\varepsilon)$. Taking without any loss of generality $x_d = 0$, we define

$$x_\eta = \frac{x}{\eta(\varepsilon)}, \quad G(x_\eta;\ \varepsilon) = F(x;\ \varepsilon) \tag{5}$$

with $\eta(\varepsilon)$ a function defined in Ξ.

Definition (Meyer, 1967). If the function $G(x_\eta;+0) = \lim G(x_\eta;\varepsilon)$, $\varepsilon \to 0$ exists uniformly on $\{x_\eta/|x_\eta| > 0\}$, then we define $\lim_\eta F(x;\varepsilon) = G(x_\eta;\ +0)$.

Thus, if $\eta \to 0$ as $\varepsilon \to 0$, then, in the limit process, $x \to 0$ also with the same speed of η, so that x_η tends to a non-zero limit value.

One of the central results of Kaplun's work is the extension theorem, which is presented here in the following version (Meyer, 1967).

Kaplun's extension theorem. If $f(x;\varepsilon)$ is a $\xi(\varepsilon)$-approximation to $g(x;\varepsilon)$ uniformly in a closed interval D_0, then it is so also in an open set $D \supset D_0$.

The above theorem was first published in Kaplun and Lagerstrom (1957) in connection with the Stokes paradox for flow at low Reynolds number. It needs to be complemented by an Axiom and by an Ansatz to relate the formal domain of validity of an equation with the actual domain of validity of its solution. The idea of Kaplun was to shift the emphasis to applying limit-processes directly to the equations rather than to the solutions, establishing some rules to determine the domain of validity of solutions from the formal domain of validity of an equation.

The set of equations that will result from passage of the limit is referred to by Kaplun as the "splitting" of the differential equations. The splitting must be seen as a formal property of the equation obtained through a "formal passage of the η-limit process." To every order of η a correspondence is induced, $\lim_\eta \to$ associated equation, on that subset of Ξ for which the associated equation exists.

Definition. The formal limit domain of an associated equation E is the set of orders η such that the η-limit process applied to the original equation yields E.

Passage of the η-limit will give equations that are distinguished in two ways: (1) they are determined by specific choices of η; and (2) they are more complete, or in Kaplun's words, "richer" than the others, in the sense that application of the η-limit process to them will result in other associated equations, but neither of them can be obtained from any of the other equations.

Limit processes that yield "rich" equations are called principal limit processes. The significance of principal limit processes is that the resulting equations are expected to be satisfied by the corresponding limits of the exact solution. The notion of principal equation is formalized below.

The above concepts and ideas can be given a more rigorous interpretation if we introduce Kaplun's concept of equivalent in the limit for a given set of equations for a given point (η, δ) of the $(\Xi \times \Sigma)$ product space.

Given any two associated equations E_1 and E_2, we define the remainder of E_1 with relation to E_2 as

$$\mathbb{R}(x_\eta;\ \varepsilon) = E_1(x_\eta;\ \varepsilon) - E_2(x_\eta;\ \varepsilon) \tag{6}$$

where ε denotes a small parameter.

According to Kaplun (1967), $\mathbb{R}$ should be interpreted as an operator giving the "apparent force" that must be added to E_2 to yield E_1.

Definition (of equivalence in the limit) (Kaplun, 1967). Two equations E_1 and E_2 are said to be *equivalent in the limit* for a given limit process, $\lim_\eta$, and to a given order, δ, if

$$\frac{\mathbb{R}(x_\eta;\ \varepsilon)}{\delta} \to 0, \text{ as } \varepsilon \to 0, \ x_\eta \text{ fixed} \tag{7}$$

The following propositions are important; they can be found in Kaplun (1967). The symbol $\sim$ is used to indicate equivalent in the limit, whereas $\nsim$ indicates not equivalent in the limit.

Proposition 1: If $E \sim E'$ for the point (η', δ') of the product space $\Xi \times \Sigma$, then $E \sim E'$ for all points (η, δ) such that $\eta = \eta'$ and $\delta >> \delta'$. Conversely, if $E \nsim E'$ for the point (η', δ'), then $E \nsim E'$ for all points (η, δ) such that $\eta = \eta'$ and ord $\delta <<$ ord δ'.

Proposition 2: If $E \sim E'$ for the point (η, δ) of the product space $\Xi \times \Sigma$, and if associated equations for that point exist for E, then they exist also for E' and are identical for both.

Proposition 3: If associated equations exist for E and E', respectively, corresponding to $\eta = \eta'$ and the sequence $\delta = \delta'_0, \delta'_1, \ldots, \delta'_n, \delta'$ where $\delta'_n > \delta' > \delta'_{n+1}$, and are identical for both, then $E \sim E'$ for the point (η', δ').

We can make the following definition.

Definition (of formal domain of validity). The formal domain of validity to order δ of an equation E of formal limit domain D is the set $D_e = D \cup D_i's$, where $D_i's$ are the formal limit domains of all equations E_i' such that E and E_i' are equivalent in D_i' to order δ.

Definition (of principal equation). An equation E of formal limit domain D is said to be principal to order δ if: (1) one can find another equation E', of formal limit domain D', such that E and E' are equivalent in D' to order δ; and (2)E is not equivalent to order δ to any other equation in D.

An equation that is not principal is said to be intermediate.

To relate the formal properties of equations to the actual problem of determining the uniform domain of validity of solutions, Kaplun (1967) advanced two assertions, the Axiom of Existence and the Ansatz about domains of validity. These assertions constitute primitive and unverifiable assumptions of perturbation theory.

Axiom (of existence) (Kaplun, 1967). If equations E and E' are equivalent in the limit to the order δ for a certain region, then given a solution S of E, which lies in the region of equivalence of E and E', there exists a solution S' of E' such that as $\varepsilon \to 0$, $|S - S'|/\delta \to 0$, in the region of equivalence of E and E'.

In other words, the axiom states that there exists a solution S' of E' such that the "distance" between S and S' is of the same order of magnitude as that between E and E'.

In using perturbation methods, the common approach is to consider the existence of certain limits of the exact solution or expansions of a certain form. This is normally a sufficient condition to find the associated equations and to assure that the axiom is satisfied (Kaplun [1967]). Equivalence in the limit, however, is a necessary condition as shown by propositions (1) to (3).

To the axiom of existence there corresponds an Ansatz, namely, that there exists a solution S of E, which lies in the region of equivalence of E and E'. More explicitly, we write.

Ansatz (about domains of validity) (Kaplun, 1967). An equation with a given formal domain of validity D has a solution whose actual domain of validity corresponds to D.

The word "corresponds to" in the Ansatz was assumed by Kaplun to actually mean "is equal to"; this establishes the link we needed between the "formal" properties of the equation and the actual properties of the solution.

The Ansatz can always be subjected to a *canonical test*, which consists in exhibiting a solution S' of E' that lies in the region of equivalence of E and E' and is determined by the boundary conditions that correspond to S.

Because of the heuristic nature of the Axiom and of the Ansatz, comparison to experiments will always be important for validation purposes. The theory, however, as implemented through the above procedure, is always helpful in understanding the matching process and in constructing the appropriate asymptotic expansions.

THE PROPOSITION OF THE PRINCIPAL EQUATIONS

The "splitting" of the equations obtained through the definition of equivalent in the limit may be extended to higher orders by introducing a fictitious perturbation of an arbitrary order δ. Thus, according to Kaplun (1967), for higher orders the splitting of the equations corresponding to arbitrary limit processes becomes more complicated and less significant; the operation of splitting is then merely reduced to exhibit some of the typical associated equations and some of the sufficient conditions under which they are associated. In fact, Kaplun lists three reasons why the splitting for higher orders should not be considered in detail: (1) the equations associated with a given point (η, δ) depend on the choice of the δ'_n for the corresponding limit process and may depend on the amount of information used in connection with the preceding terms, (2) the δ'_n depend to a greater extent on boundary conditions and hence are difficult to determine a priori, and (3) much trivial splitting of the associated equations arises, corresponding to expansions of the preceding terms by different limit proccsses.

Here, we want to further extend the above notions. In what follows we will show that, for certain points of the (Ξ, Σ) product space, the determination of the associated equations will depend on the choice of some discrete values of δ'_n. Results indicate that the order of validity of an approximation is defined by open intervals determined by the discrete δ_n's. Furthermore, no trivial splitting results in these certain points.

To extend the previous results to higher orders, we consider solutions of the form

$$f = f_0 + \Delta(\varepsilon) f_1 \tag{8}$$

where $\Delta(\varepsilon) \in \Xi$.

Some questions are now in order. Which function is $\Delta(\varepsilon)$ for a given differential equation? Is $\Delta(\varepsilon)$ the same for all regions of the domain?

The first question is complex and involves speculating on the existence and uniqueness of solutions. Of course, uniqueness of $\Delta(\varepsilon)$ can never be assured because given any $\Delta(\varepsilon)$, one can always present another $\Delta'(\varepsilon)$, such that $\Delta'(\varepsilon)$ is exponentially close to $\Delta(\varepsilon)$. Thus, according to Kaplun, there will always be a "question of choice" for the determination of the appropriate asymptotic expansions that must be solved relying on intuition and physical insight. An adequate $\Delta(\varepsilon)$ can, however, be determined in a very natural way. We require $\Delta(\varepsilon)$ to be such that the resulting equation for f_1 does not provide a trivial solution. A $\Delta(\varepsilon)$ satisfying this condition is said to be a critical $\Delta(\varepsilon)$. Analogously, its order, ord $\Delta(\varepsilon)$, is called critical order. More precisely:

Definition (of critical order) (Kaplun [1967]). An order ord $\Delta(\varepsilon)$ is said to be critical if: (1) the corrections to f_0 to any order ζ in D, $D = \{\zeta/\text{ord } \Delta(\varepsilon < \text{ord } \zeta < 1\}$ are trivial; and (2) the corrections to f_0 to any order ζ in the complement of D are not trivial.

The above definition suggests that approximate solutions for different regions of the domain should not in general have the same $\Delta(\varepsilon)$. Of course, equal $\Delta's$ might happen as a mere coincidence; however, it is important to give emphasis to the fact that, normally, this is not the case.

To find the several order approximate equations we substitute Eq. (8) into the original equations and perform elementary operations such as addition, multiplication, subtraction, differentiation, and so on. If these operations are justified, that is, if they do not lead to any nonuniformity, we then collect the terms of same order of magnitude and construct a set of approximate equations. Thus, it is clear that in the process of collecting terms, to each term E_1 of order, say ν, there will always correspond another term E_2 of order $\nu\Delta(\varepsilon)$.

Consider now an equation E where E_1 and E_2 denote the first two critical order terms. We call the operator $\Pi_1(E) = E_1$ the first-order projection of E onto E_1. Analogously, the operator $\Pi_2(E) = E_2$ is called the second-order projection of E onto E_2.

We can then enunciate the following proposition.

Proposition (of the principal equations). If there is a principal equation, E_1, at a point $(\eta, 1)$ of the (Ξ, Σ) product space, then there is also a principal equation, E, at a point (η, ε) of (Ξ, Σ) with $E_1 = \Pi_1(E)$.

Proof: Suppose E_1 is a principal equation at a point $(\eta, 1)$ of the (Ξ, Σ) product space. Then one can find a term R_{1l} such that R_{1l} is order unity in D (the domain of E_1) but ord $1 <$ ord $R_{1l} <$ ord ε in D_u, the upper adjacent domain of D. Here ε denotes the first critical order. Define $E_1' = E_1 - R_{1l}$.

Let E_2 and E_2' now denote the first-order associated equations in D and D_u, respectively. Then, there is a term R_{2l} such that R_{2l} is order ε in D but ord $R_{2l} <$ ord ε in D_u. Define $E_2' = E_2 - R_{2l}$.

As a result, the structure of the lower adjacent region is

$$\text{ord } R_{1l} < \text{ord } E_1' < \text{ord } R_{2l} < \text{ord } E_2' \tag{9}$$

This shows that no other equation is equivalent to order ε to equation $E = E_1' + R_{1l} + E_2' + R_{2l}$ in D. However, E and $E'(= E_1' + R_{1l} + E_2')$ are equivalent in D_u to ord ε. We conclude that E is a principal equation at a point (η, ε) of the (Ξ, Σ) product space.

The converse of the above proposition is obviously true, that is, if E is a principal equation at a point (η, ε) of the (Ξ, Σ) space, where η denotes the formal limit domain of E and ε the first critical order, then $\prod_1(E)$ is a principal equation at a point $(\eta, 1)$ of the (Ξ, Σ) space.

What the above proposition clearly states is that the position in the (Ξ, Σ) product space where the principal equations are located can be searched by looking only at the lowest-order associated equations. Furthermore, it says that these lowest-order approximations are good up to the first critical order and that no trivial splitting will arise. This fact is only valid for the particular point in (Ξ, Σ) space where the principal equation holds. In the upper and lower adjacent domains trivial splitting occurs.

Results indicate that higher order splitting should not, in fact, be considered. The principal equations of the problem, those that retain most of the information about the problem solution, can have their position determined only through an analysis of the lowest order terms. Then the concept of critical order can be applied to the solution to find the appropriate asymptotic expansions for the problem.

THE ASYMPTOTIC STRUCTURE OF THE TURBULENT BOUNDARY LAYER

Boundary layer problems are historically important in the development of singular perturbation methods. In fact, the basic ideas of singular perturbation methods can be traced back to Prandtl's boundary layer theory of a laminar flow. Prandtl's matching principle for laminar boundary layers was systematically discussed and generalized in the 1950s yielding well-established procedures and solutions that have rendered the laminar flow problem solved.

In the 1970s, the interest shifted to turbulent flow. Two approaches were used: in the first, asymptotic techniques were applied to the averaged equations without appealing to any closure model (Yajnik [1970], Mellor [1972]); in the second, eddy-viscosity (Bush and Fendell [1972]) or $\kappa - \varepsilon$ (Deriat and Guiraud [1986]) models were used to find high-order approximations. The theories divided the turbulent boundary layer into two regions, becoming the basis for most subsequent work. Other authors, Long and Chen (1981), Sychev and Sychev (1987), and Melnik (1989), however, recently have claimed that the turbulent boundary layer instead has a three-layered structure. This structure considers a new region in which a balance of inertia forces, and pressure and turbulent friction forces occur. The formulation of Melnik is based on a two-parameter expansion of the boundary layer equations, the new additional small parameter resulting from the particular turbulence closure model he uses.

The discussions that have led to the development of the three-layered asymptotic model for the turbulent boundary layer result from the recognition that two-layered models cannot deal with large flow disturbances in the stream-wise direction. When a turbulent boundary layer is subjected to a large longitudinal adverse pressure gradient, the velocity deficit is large and the mean momentum equation is nonlinear; this makes the classic matching arguments that result in a log-law and in a two-deck structure not valid anymore. The classic wall characteristic velocity, the friction velocity, may become an inappropriate scaling parameter so that new formulations will have to be developed for the problem at hand.

Here, we investigate the turbulent boundary layer from the point of view of Kaplun's single limits. The purpose is to formally arrive at a three-layered structure that is compatible with the class of problems to be studied: the zero-pressure gradient turbulent boundary layer, the separating turbulent boundary layer, and the shock-wave interacting turbulent boundary layer.

The Model Equation

In this work, the asymptotic structure of the turbulent boundary layer is investigated by applying Kaplun limits to a model equation that mimics turbulent flow passed over a flat surface. The model equation was chosen for being relatively simple while retaining most of the features of the real problem and hopefully will aid further understanding of its conceptual structure. One can easily grasp the main concepts and ideas without being hampered by unnecessary details. It turns out that a two-layered structure is derived, which, however, does not attain the length scales of Sychev and Sychev. A skin-friction equation is obtained from the matching conditions, and the level of the logarithmic part of the solution, plotted in appropriate coordinates, is shown to be –1.325. This constant results from the type of turbulence model used here, being different from the real turbulent boundary layer problem constant (= 5).

Consider the partial differential equation

$$f\frac{\partial f}{\partial x} - \frac{\partial}{\partial y}\left(\nu\frac{\partial f}{\partial y}\right) - \varepsilon\frac{\partial^2 f}{\partial y^2} = 0 \tag{10}$$

$$\nu = \begin{cases} \varkappa^2 y^2 \frac{\partial f}{\partial y} & \text{if } y < y_1 \\ \alpha\varepsilon\, x f_0\, f & \text{if } y \geq y_1 \end{cases} \tag{11}$$

subject to the boundary conditions

$$x = 0, \qquad f = 1 \tag{12}$$

$$y = 0, \qquad f = 0 \tag{13}$$

$$y \to \infty, \qquad f = 1 \tag{14}$$

and where y_1 = point where continuity of f, $\partial f/\partial y$, and ν must be ensured, $\varepsilon = 10^{-6}$, $\alpha = 0.02$, $\varkappa = 0.4$, and $f_0(x) = \partial f(x,0)/\partial y$.

The inertia term in the Navier-Stokes averaged equations is represented here by $f\partial f/\partial x$. For the turbulence term, the middle term in Eq. (10), a simple mixing-length/eddy viscosity model is assumed so that the complexity of the problem can be kept to a manageable level, and analytical solutions can be obtained. The term with the second derivative in y represents the diffusion term with ε playing the role of the inverse of the Reynolds number. The boundary conditions are standard; thus, at the wall ($y = 0$), the no-slip condition has to be satisfied, whereas far away from it the solution must approach a uniform external flow value.

To analyze the problem defined by Eq. (10) we consider the following stretching transformations

$$x_\Delta = \frac{x}{\Delta(\varepsilon)}, \quad y_\eta = \frac{y}{\eta(\varepsilon)}, \quad g(x_\Delta, y_\eta) = f(x, y) \tag{15}$$

with $\Delta(\varepsilon)$ and $\eta(\varepsilon)$ defined on Ξ.

The resulting transformed equation is

$$\frac{g}{\Delta(\varepsilon)}\frac{\partial g}{\partial x_\Delta} - \frac{\varepsilon}{\eta(\varepsilon)^2}\frac{\partial^2 g}{\partial y_\eta^2} - \frac{\partial}{\partial y_\eta}\left(\nu_\eta \frac{\partial f}{\partial y_\eta}\right) = 0 \tag{16}$$

Passing the η-limit onto Eq. (16) one gets

$$\text{ord}\, A\,\Delta < \text{ord}\,\eta < 1: \quad g\frac{\partial g}{\partial x_\Delta} = 0 \tag{17}$$

$$\text{ord}\, A\Delta = \text{ord}\,\eta: \quad g\frac{\partial g}{\partial x_\Delta} - \frac{\partial}{\partial y_\eta}\left(\nu_\eta \frac{\partial f}{\partial y_\eta}\right) = 0 \tag{18}$$

$$\text{ord}\ \varepsilon/A < \text{ord}\,\eta < \text{ord}\ A\Delta: \quad \frac{\partial}{\partial y_\eta}\left(\nu_\eta \frac{\partial f}{\partial y_\eta}\right) = 0 \tag{19}$$

$$\text{ord}\ \varepsilon/A = \text{ord}\,\eta: \quad \frac{\partial^2 g}{\partial y_\eta^2} - \frac{\partial}{\partial y_\eta}\left(\nu_\eta \frac{\partial f}{\partial y_\eta}\right) = 0 \tag{20}$$

$$\text{ord}\,\eta < \text{ord}\ \varepsilon/A: \quad \frac{\partial^2 g}{\partial y_\eta^2} = 0 \tag{21}$$

The above equations were derived under the assumption that the derivatives are ord(A). The significance of A with its physical interpretation is given in the following text.

Equations (18) and (20) are the principal equations. A complete solution to our problem should then, according to the Axiom of Existence and Kaplun's Ansatz, be obtained from these equations. The formal domains of validity of these equations cover the entire domain and overlap in

$$D_0 = \{\text{ord } \eta \text{ such that ord } \varepsilon/A < \text{ord } \eta < \text{ord } A\Delta\} \tag{22}$$

The solution of the inner equation, Eq. (20), is given by

$$g_i = \frac{A}{\varkappa}\left[\frac{1 - \sqrt{\zeta^2 + 1}}{\zeta} + \ln\left(\zeta + \sqrt{1 + \zeta^2}\right)\right] \tag{23}$$

with $A = A(x)$ and $\zeta = 2A\varkappa y/\varepsilon$.

To find a solution for the outer region equation, Eq. (18), when $y < y_1$, we introduce the variable transformation $t = y_\eta/x_\Delta = y/Ax$ and obtain

$$th'' + h' + \frac{1}{2\varkappa^2}h = 0 \tag{24}$$

$h(t) = g(x_\Delta, y_\eta)$.

The above equation does not have a global analytical solution in domain ord $\eta >$ ord ε/A; however, it is possible to find the local behavior that the solution exhibits in the neighborhood of the regular singular point $t = 0$. Thus, as $t \to 0$, we find

$$h = 1 + B \ln t - \frac{t}{2\varkappa^2} - \frac{Bt}{2\varkappa^2}\ln t + \frac{t^2}{16\varkappa^4} + \frac{Bt}{\varkappa^2} + \frac{B}{16\varkappa^4}t^2 \ln t$$
$$+ \left[\frac{1}{36} + \frac{B}{4}\right]\frac{t^2}{8\varkappa^6} - \frac{9B}{16\varkappa^2}t^2 + \cdots \tag{25}$$

On the other hand, when $y \geq y_1$, Eq. (18) becomes

$$tp' + \alpha p'' + \alpha p'^2 = 0 \tag{26}$$

where $p(t) = g(x_\Delta, y_\eta)$, $t = y_\eta/x_\Delta = y/Ax$.

The nonlinear equation above does not have a general solution. To find an approximate solution we make

$$p(t) = 1 - q(t) \tag{27}$$

and consider $q, q' << 1$.

Then, Eq. (26) becomes

$$tq' + \alpha p'' = 0 \tag{28}$$

and, as a result, p can be written as,

$$p = 1 + E + D\int_{t_1}^{t} e^{-t^2/2\alpha} dt \tag{29}$$

where D and E are constants to be determined.

To find a solution valid in the whole domain we must show Eqs. (23) and (25) to be equivalent in an overlap domain, that is, we must show

$$\lim_\eta (g_i(x_\Delta, y_\eta) - h(y_\eta/x_\Delta)) = 0 \tag{30}$$

η belonging to (22).

From the above limit process, we find

$$\frac{A}{\varkappa}\ln\frac{4\varkappa Ay}{\varepsilon} - \frac{A}{\varkappa} = C + B\ln\frac{y\Delta}{x\eta} \tag{31}$$

This equation shows that overlapping can only be achieved if

$$B = \frac{A}{\varkappa} \tag{32}$$

and

$$\frac{A}{\varkappa}\ln\frac{4\varkappa A\eta x}{\varepsilon\Delta} - \frac{A}{\varkappa} = C \tag{33}$$

A close inspection of Eq. (33) reveals that in the limit as $\varepsilon \to 0$

$$\text{ord}\,(A \ln \varepsilon^{-1}) = \text{ord}\ 1 \tag{34}$$

and hence

$$\text{ord}\ A = \text{ord}\left(\ln \varepsilon^{-1}\right)^{-1} \ll \text{ord}\ 1 \tag{35}$$

This means that the outer and inner regions stretching variables are indeed those indicated in Eqs. (18) and (20), that is,

$$\text{ord}\ \eta = \text{ord}\ (A\Delta) \tag{36}$$

and

$$\text{ord}\ \eta = \text{ord}\ (\varepsilon/A) \tag{37}$$

respectively.

Because in a turbulent boundary layer the wall stress is given by

$$\tau_w = \mu\frac{\partial u}{\partial y}(x, 0) = \varepsilon\frac{\partial f}{\partial y}(x, 0) = A^2 \tag{38}$$

parameter A represents the friction velocity.

Solutions (23) and (25) can be combined through a composite function rule, that is, by making

$$g_c = g_i + h - \lim_\eta(g_i(x_\Delta, y_\eta) - h(y_\eta/x_\Delta)) \tag{39}$$

This procedure yields

$$g_c = \frac{A}{\varkappa}\left\{\frac{1-\sqrt{\zeta^2+1}}{\zeta} + \ln\left(\zeta + \sqrt{1+\zeta^2}\right)\right\} - \frac{t}{2\varkappa^2} - \frac{At}{2\varkappa^3}\ln t + \frac{t^2}{16\varkappa^4}$$
$$+ \frac{At}{\varkappa^3} + \frac{A}{16\varkappa^5}t^2\ln t + \left[\frac{1}{36} + \frac{A}{4\varkappa}\right]t^3 - \frac{9A}{16\varkappa^3}t^2 \tag{40}$$

where $t = y/Ax$.

The constants in Eq. (28) can be evaluated by considering the boundary condition at infinity and the continuity of solution at point $y = y_1$.

The continuity condition,

$$g_c(y_1, x) = p(y_1/Ax), \qquad y = y_1 \tag{41}$$

implies that

$$E = g_c(y_1A/\varepsilon) - 1 \tag{42}$$

The boundary condition, $y \to \infty, f = 1$, gives

$$D = \frac{1 - E}{\sqrt{\frac{\alpha\pi}{2}}\left(1 - \mathrm{erf}\left(\frac{y_1}{\sqrt{2\alpha Ax}}\right)\right)} \tag{43}$$

The continuity condition applied to v, that is,

$$\alpha\varepsilon f_0 xp\Big|_{t=y_1/Ax} = \left(\varkappa^2 y^2\left(\frac{\partial g_c}{\partial y}\right)\right)\Bigg|_{\zeta=\frac{2\varkappa Ay_1}{\varepsilon}} \tag{44}$$

results that

$$y_1 = \frac{\alpha Ax}{\varkappa} \tag{45}$$

Finally, continuity of the derivatives,

$$\frac{\partial}{\partial y} g_c(y_1, x) = \frac{\partial}{\partial y} p(y_1/Ax) \tag{46}$$

furnishes,

$$\frac{A}{\varkappa} \ln \frac{4\alpha A^2 x}{\varepsilon} - \frac{A}{\varkappa} = 1 - A\sqrt{\frac{2}{\alpha}}\left[1 - \operatorname{erf}\left(\frac{1}{\varkappa}\sqrt{\frac{\alpha}{2}}\right)\right]\frac{\sqrt{\pi}}{2}\exp\left(-\frac{1}{\varkappa}\sqrt{\frac{\alpha}{2}}\right) \tag{47}$$

The solution of the above transcedental algebraic system furnishes the unknown parameter A. Equation (47) can easily be recognized as the counterpart of the skin-friction equation for an incompressible flow. An equation similar to Eq. (45) for the determination of the patching point for the outer solutions, that is, for the patching of the outer region turbulence models, however, cannot be found in literature. This point has to be determined through sucessive interactions of a numerical solution.

The values of A and y_1 obtained at various stations are shown in Table 1. We recall that in physical terms parameter A represents the friction at the wall.

The agreement between the present procedure and the "exact" numerical solution is remarkable.

Comparison of the analytical results given by equation

$$f = \begin{cases} g_c(x, y) & \text{if } y < y_1; \\ p(x/Ay) & \text{otherwise} \end{cases} \tag{48}$$

with a numerical solution is shown in Fig. 1.

Equation (10) was solved numerically through an implicit finite difference scheme. Centered differences accurate to second order are used to discretize the transversal derivatives, whereas backward differences accurate to first order are used to discretize the longitudinal derivatives. Simple lagging is used to linearize the obtained algebraic equations. The resulting linearized system is solved by the Thomas algorithm and iteration is used to handle the nonlinearity, with the position of y_1 being reevaluated each time. Only three iterations were shown to be necessary for convergence of the derivative of f at the wall to an absolute error of 10^{-5}.

To estimate the overlap domain of Eqs. (18) and (20) we use Kaplun's concept of equivalent in the limit to obtain

$$\mathbb{R} = \frac{\frac{\varepsilon}{\eta^2}\frac{\partial^2 g}{\partial y_\eta^2} - \frac{1}{\Delta} g \frac{\partial g}{\partial x_\Delta}}{\varepsilon^\gamma} \tag{49}$$

Table 1 Predictions of A and y_1

	Theoretical		Numerical	
x	A	y_1	A	y_1
5	0.0446	0.0115	0.0440	0.0112
10	0.0419	0.0209	0.0416	0.0206
15	0.0405	0.0304	0.0403	0.0300
20	0.0395	0.0395	0.0394	0.0392

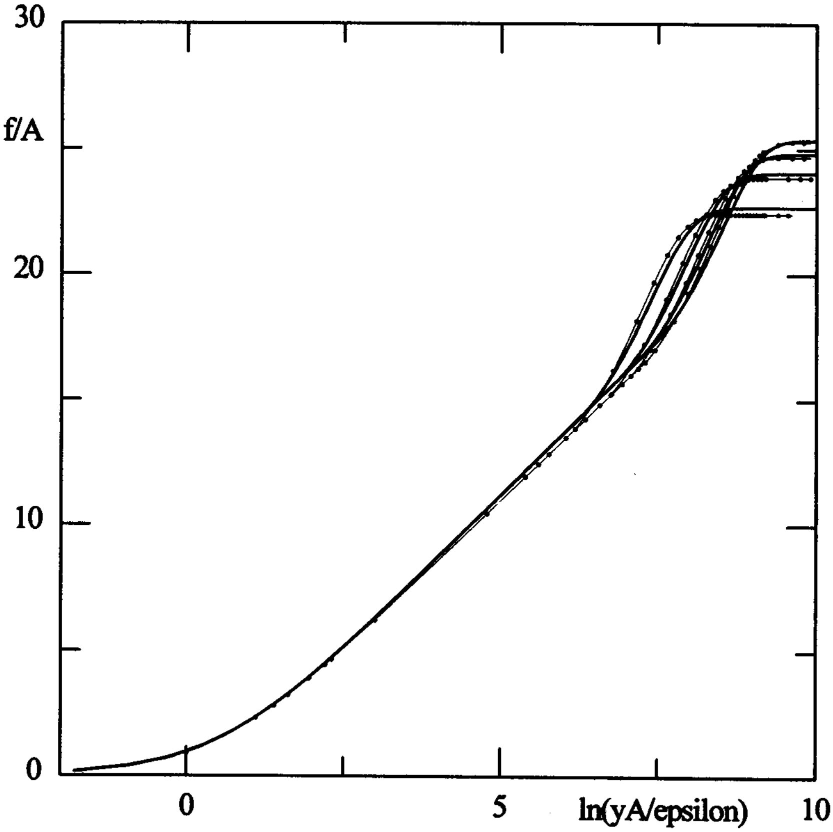

Fig. 1 Comparison of results: solid lines, numerical; dotted lines, analytical. Stations: $x = 5, 10, 15, 20$.

Noting that the leading order term in region $\varepsilon/A < \eta < A\Delta$ is the turbulent term, of ord (A^2/η), we normalize the above equation to order unity to find

$$\mathbb{R} = \frac{\gamma}{A^2\varepsilon^\gamma}\left[\frac{\varepsilon}{\eta^2}\frac{\partial^2 g}{\partial y_\eta^2} - \frac{1}{\Delta}g\frac{\partial g}{\partial x_\Delta}\right] \tag{50}$$

Then because ord $(\partial^2 g/\partial y_\eta^2)$ = ord $(g\partial g/\partial x_\Delta) = A$, the formal domain of overlapping is given by

$$D = \{\eta \text{ such that ord } (\varepsilon^{1-\gamma}/A) < \text{ord } \eta < \text{ord } (\varepsilon^\gamma A\Delta)\} \tag{51}$$

Thus, according to Kaplun's Ansatz about domains of validity, the above approximate solutions, Eqs. (23) and (25) only overlap if

$$0 \le \gamma \le \frac{1}{2}\left(1 - \frac{\ln A^2\Delta}{\ln \varepsilon}\right) \tag{52}$$

The application of Kaplun limits to our model equation does suggest the turbulent boundary layer to have a two-deck structure, the principal equations being located at points $A\Delta$ and ε/A of the Ξ space. For flows where an additional pressure-gradient term must be considered in the model equation, the asymptotic structure will remain the same, but the position of the outer principal equation will change to $A^2\Delta$.

The analytic solution shows that, in the overlap region, the inner equation tends asymptotically to

$$g_i = \frac{A}{\varkappa} \ln \frac{4A\varkappa y}{\varepsilon} - \frac{A}{\varkappa} \tag{53}$$

that is,

$$g_i = \frac{A}{\varkappa} \left(\ln \frac{Ay}{\varepsilon} - 1.325 \right) \tag{54}$$

an expression different from the law of the wall equation commonly quoted in literature.

For real flows, the linear coefficient in Eq. (53) is 5.0 yielding values of A lower than those found in the present analysis. The constant −1.325 in Eq. (53) is solely a consequence of the specified turbulence model, which must be modified if a more realistic description of the flow is desired.

To the author's knowledge, this is the first time that Kaplun limits are used to analyze the turbulent boundary layer problem where quantitative results are presented. In particular, it is the first time that Eq. (40) is presented. In most analyses of the problem, approximate solutions are obtained by breaking up Eq. (20) into the two intermediate Eqs. (19) and (21). This procedure introduces a spurious constant into the problem associated with the patching of the intermediate solutions in ord η = ord ε/A, which is determined on physical grounds and hence ensures a reasonable agreement of the analytic solution with the experimental profile. Equation (40) results from the principal equation solutions, and for this reason dispenses this true juggling.

In fact, in all proposed solutions for the turbulent boundary layer, the skin-friction equation results from the matching of the inner and outer solutions. Here, the skin-friction equation was shown to result from the matching of the *derivatives* of the inner and outer solutions.

The Zero-Pressure Gradient Turbulent Boundary Layer

For an incompressible two-dimensional turbulent flow over a smooth surface in a prescribed pressure distribution, the time-averaged motion equations, that is, the continuity equation and the Navier-Stokes equation can be written as

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{55}$$

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_j} - \varepsilon^2 \frac{\partial}{\partial x_j}\left(\overline{u_j' u_i'}\right) + \frac{1}{R}\frac{\partial^2 u_i}{\partial x_j^2} \tag{56}$$

where the notation is classical. Thus, $(x_1, x_2) = (x, y)$ stand for the coordinates $(u_1, u_2) = (u, v)$ for the velocities, p for pressure, and R for the Reynolds number. The dashes are used to indicate a fluctuating quantity. In the fluctuation terms, an overbar is used to indicate a time average.

All mean variables refer to some characteristic quantity of the external flow. The velocity fluctuations, on the other hand, refer to a characteristic velocity, u_R, first introduced in Cruz and Silva Freire (1998).

The correct assessment of the characteristic velocity is fundamental for the determination of the boundary layer asymptotic structure. For unseparated flows the characteristic velocity is known to be the friction velocity; for separating flows it reduces to $(\nu(dp/dx)/\rho)^{1/3}$. For the moment, we will consider attached flow so that we can write

$$\text{ord}\left(u_i'\right) = \text{ord}\left(u_\tau\right) \tag{57}$$

This result is valid for incompressible flows as well as for compressible flows (see, e.g., Kistler [1956] and Kistler and Chen [1956]).

The small parameter ε therefore is defined by

$$\varepsilon = \frac{u_R}{U_\infty} = \frac{u_\tau}{U_\infty} \tag{58}$$

The asymptotic expansions for the flow parameters are written as

$$u(x, y) = u_1(x, y) + \varepsilon u_2(x, y) \tag{59}$$

$$v(x, y) = \eta[v_1(x, y) + \varepsilon v_2(x, y)] \tag{60}$$

$$p(x, y) = p_1(x, y) + \varepsilon p_2(x, y) \tag{61}$$

$$u_i'(x, y) = \varepsilon u_{i1}'(x, y) + \varepsilon^2 u_{i2}'(x, y) \tag{62}$$

$$\overline{u'v'}(x, y) = \varepsilon^2 \overline{u'v_1'}(x, y) + \varepsilon^3 \overline{u'v_2'}(x, y) \tag{63}$$

To find the asymptotic structure of the boundary layer we consider the following stretching transformation

$$\hat{y} = y_\eta = \frac{y}{\eta(\varepsilon)}, \quad \hat{u}_i\left(x, y_\eta\right) = u_i(x, y) \tag{64}$$

with $\eta(\varepsilon)$ defined on Ξ.

After substitution of Eq. (64) into Eqs. (59) to (62) and after passage of the η-limit process onto the resulting equation we get:

x-momentum equation:

	$\text{ord}(\delta) = 1$		$\text{ord}(\delta) = \text{ord}(\varepsilon)$	
$\eta = 1$:	$D_{11} = P_1,$		$D_{12} + D_{21} = P_2$	(65)
$\varepsilon < \eta < 1$:	$D_{11} = P_1,$		$D_{12} + D_{21} = P_2$	(66)
$\eta = \varepsilon$:	$D_{11} = P_1,$		$D_{12} + D_{21} = P_2 - \left(\overline{u'v'_1}\right)_{\hat{y}}$	(67)
$\varepsilon^2 < \eta < \varepsilon$:	$D_{11} = P_1,$	$\left(\overline{u'v'_1}\right)_{\hat{y}} = 0,$	$D_{12} + D_{21} = P_2$	(68)
$\eta = \varepsilon^2$:	$D_{11} = P_1 - \left(\overline{u'v'_1}\right)_{\hat{y}},$		$D_{12} + D_{21} = P_2 - \left(\overline{u'v'_2}\right)_{\hat{y}}$	(69)
$\varepsilon^3 < \eta < \varepsilon^2$:	$\left(\overline{u'v'_1}\right)_{\hat{y}} = 0,$	$D_{11} = P_1,$	$\left(\overline{u'v'_2}\right)_{\hat{y}} = 0$	(70)
$\eta = \varepsilon^3$:	$\left(\overline{u'v'_1}\right)_{\hat{y}} = 0,$		$D_{11} = P_1 - \left(\overline{u'v'_2}\right)_{\hat{y}}$	(71)
$1/\varepsilon^2 R < \eta < \varepsilon^3$:	$\left(\overline{u'v'_1}\right)_{\hat{y}} = 0,$		$\left(\overline{u'v'_2}\right)_{\hat{y}} = 0$	(72)
$\eta = 1/\varepsilon^2 R$:	$\left(\overline{u'v'_1}\right)_{\hat{y}} = 0,$		$\left(\overline{u'v'_2}\right)_{\hat{y}} = \left(u_2\right)_{\hat{y}\hat{y}}$	(73)
$1/\varepsilon R < \eta < 1/\varepsilon^2 R$:	$\left(\overline{u'v'_1}\right)_{\hat{y}} = 0,$	$\left(u_2\right)_{\hat{y}\hat{y}} = 0,$	$\left(\overline{u'v'_2}\right)_{\hat{y}} = 0$	(74)
$\eta = 1/\varepsilon R$:	$\left(\overline{u'v'_1}\right)_{\hat{y}} = \left(\hat{u}_2\right)_{\hat{y}\hat{y}},$		$\left(\overline{u'v'_2}\right)_{\hat{y}} = \left(\hat{u}_3\right)_{\hat{y}\hat{y}}$	(75)

where the following operators were used

$$D_{ij} = \hat{u}_i \frac{\partial \hat{u}_j}{\partial x} + \hat{v}_i \frac{\partial \hat{u}_j}{\partial y_\eta}, \qquad P_i = -\frac{1}{\rho}\frac{\partial \hat{p}_i}{\partial x} \tag{76}$$

The above equations were arranged in three columns according to their respective order of approximation. The first column corresponds to the first order of approximation; the third one to the second order of approximation. The middle column corresponds to orders between the first and the second critical order. The extreme left of the lines indicates the point in the domain where the η-limit process was applied.

Passage of the η-limit process onto the y-momentum equation does not give any relevant information. In fact, we will find that for ord $\eta <$ ord ε the first- and second-order pressure terms will dominate all the other terms. All information regarding the asymptotic structure of the boundary layer therefore is contained in the x-momentum equation.

The term $\hat{u}_1(x, y_\eta)$ is missing from Eqs. (74) and (75) because from the no-slip condition $\hat{u}_1 = 0$ near the wall.

Equations (69) and (75) are distinguished in two ways: (1) they are determined by specific choices of η; and (2) they are "richer" than the others in the sense that application of the limit process to them yields some of the other equations, but neither of them can be obtained from passage of the limit process to any of the other equations. Thus, according to the definitions introduced in the previous sections, these equations are the principal equations. We have seen that principal equations are important because they are expected to be satisfied by the corresponding limits of the exact solution.

A complete solution to the problem should then, according to the Axiom of Existence and Kaplun's Ansatz, be obtained from the principal equations located at points ord η = ord ε^2 and ord η = ord $(1/\varepsilon R)$. The formal domains of validity of these equations cover the entire domain and overlap in a region determined according to the definition of equivalent in limit.

To find the overlap region of Eqs. (69) and (75), we must show these equations to have a common domain where they are equivalent. A direct application of the definition of equivalence in the limit to Eqs. (69) and (75) yields

$$\mathbb{R} = \frac{D(\hat{u}_1) - P(\hat{p}_1) + D(\hat{u}_2) - P(\hat{p}_2) - (\hat{u}_2)_{\hat{y}\hat{y}} - (\hat{u}_3)_{\hat{y}\hat{y}}}{\varepsilon^\alpha} \tag{77}$$

Noting that the leading order term in region ord $(1/\varepsilon R)$ < ord η < ord ε^2 is the turbulent term of ord (ε^2/η), we normalize the above equation to order unity to find

$$\overline{\mathbb{R}} = \frac{\eta}{\varepsilon^2}\mathbb{R} \tag{78}$$

The overlap domain is the set of orders such that the η-limit process applied to $\overline{\mathbb{R}}$ tends to zero for a given α. Then because ord $(\partial/\partial y) = \varepsilon$ and ord $(\partial/\partial x) = 1$, the formal overlap domain is given by

$$D_{\text{overlap}} = \left\{\eta/\ \text{ord}\left(\varepsilon^{1+\alpha}R\right)^{-1} < \text{ord}\,\eta < \text{ord}\left(\varepsilon^{2+\alpha}\right)\right\} \tag{79}$$

According to Kaplun's Ansatz about domains of validity, the approximate equations, Eqs. (69) and (70), only overlap if set (79) is a non-empty set, that is, if

$$0 \le \alpha \le -\frac{1}{2}\left(\frac{\ln R}{\ln \varepsilon} + 3\right) \tag{80}$$

The implication is that the two-deck turbulent boundary layer structure given by the two principal equations, Eqs. (69) and (75), provides approximate solutions that are accurate to the order of $\varepsilon^{\alpha_{max}}$, where α_{max} is the least upper bound of the interval (80). This fundamental result can only be reached through the application of Kaplun's concepts and ideas to the problem.

In particular, the overlap domain of the first and second order of approximation are given, respectively, by

$$D_1^o \cap D_1^i = \left\{\text{ord}\eta/\text{ord}(1/\varepsilon R) < \text{ord}(\eta) < \text{ord}(\varepsilon^2)\right\} \tag{81}$$

and

$$D_2^o \cap D_2^i = \{\text{ord}\eta / \text{ord}(1/\varepsilon^2 R) < \text{ord}(\eta) < \text{ord}(\varepsilon^3)\} \tag{82}$$

We conclude that the turbulent boundary layer has a two-deck structure very much like the one derived by Sychev and Sychev. This structure, however, must change as a separation point is approached. We shall see this next.

Before we move forward, however, some comments about the intermediate equations will be made.

For the formal limit domains that are not adjacent to the principal equations, two approximated equations are always defined, separated by the first two critical orders. In this case the interpretation is simple and the local approximated equations and solutions well defined. For the regions adjacent to the principal equations, however, a correction with order between the first two critical orders is found. The interpretation of these equations is more complex and must be made on an individual basis. For example, in the turbulent boundary layer problem under consideration, the solution in the upper adjacent region must take into consideration, as the first two orders of approximation equations, the leading order equation and the intermediate order equation; these equations will provide nontrivial solutions with physical information. For the lower adjacent region, however, the intermediate order equation provides a trivial solution; thus, no extra information is obtained from this equation except that the overlap domain for the first two orders of approximation is not given by Eq. (82) but by

$$D_2^o \cap D_2^i = \{\text{ord}\eta / \text{ord}(1/\varepsilon^2 R) < \text{ord}\,(\eta) < \text{ord}\,(\varepsilon^2)\} \tag{83}$$

The Separating Turbulent Boundary Layer

The above asymptotic structure must undergo some modifications if flows subjected to adverse pressure gradients are to be considered (Cruz and Silva Freire, 1998).

A major difficulty found for a direct translation of the classic models into models that apply for separating flows is the characteristic velocity used in the former approach. When the friction velocity, u_τ, is used to develop the asymptotic structure of the boundary layer, a nonuniformity will occur near a separation point where $u_\tau = 0$. These difficulties force into the adverse pressure gradient problem a new small parameter of the order of $R^{-1/3}$, which is used to scale a power-y layer that replaces the logarithmic layer. This new intermediate layer defines a third characteristic scale that must be considered together with the wall and defect layer characteristic scales. Thus, three sets of characteristic scales are needed for the asymptotic description of adverse pressure gradient turbulent boundary layers (see Durbin and Belcher [1992]).

The result is that any theory advanced for the problem should explain in asymptotic terms how the far upstream two-deck structure reduces to a three-deck structure near a separation point. Equivalently, any theory should show how the logarithmic layer vanishes as separation is approached, and how the $y^{1/2}$-layer is formed.

In Cruz and Silva Freire (1998), a new scaling procedure was introduced through asymptotic arguments that resulted in an algebraic equation for u_R that yielded a changeable asymptotic structure for the boundary layer consistent with the experimental data. The theory followed the approaches of Yajnik (1970) and of Mellor (1972), not imposing any functional relationship between quantities determined by the Reynolds stress field and by the velocity field.

Here we repeat part of the theory to illustrate how the results of the previous section can be extended to separated flows.

In the region defined by the principal equation, Eq. (75), a balance between the turbulent and viscous stresses exists so that we may write

$$\frac{\partial}{\partial y}\left(\overline{-\rho u'v'}\right) + \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \tag{84}$$

In this region, the characteristic length is given by ν/u_R. Then, considering that the turbulent fluctuations are of the order of the reference velocity, u_R, and that the viscous term can be approximated by

$$\text{ord}\left(\mu \frac{\partial u_2}{\partial y}\right) = \text{ord}\left(\tau_w\right) \tag{85}$$

results from simple order of magnitude arguments show that the characteristic velocity can be estimated from the algebraic equation

$$u_R^3 - \frac{\tau_w}{\rho} u_R - \frac{\nu}{\rho}\frac{\partial p}{\partial x} = 0 \tag{86}$$

Passing the limit as τ_w tends to zero onto the above equation,

$$u_R \to \left(\frac{\nu}{\rho}\frac{\partial p}{\partial x}\right)^{1/3} \tag{87}$$

so that the characteristic velocity for the near separation point region proposed by Stratford (1959) and by Townsend (1976) is recovered.

The characteristic velocity u_R is determined by the highest real root of (86).

It follows that, close to the separation point, ord (ε^2) = ord $(1/\varepsilon R)$, and the two principal equations merge giving origin to a one-deck structure. This merging provokes the disappearance of the log-region, reducing the flow structure to a wake region and a viscous region.

To find the asymptotic structure of the separating boundary layer we apply the following stretching transformation to the equations of the previous section

$$x_\Delta = \frac{x}{\Delta(\varepsilon)} \tag{88}$$

with $\Delta(\varepsilon)$ defined on Ξ.

The resulting flow structure is given by:

x-momentum equation:

$$\text{ord}\,\Delta = \text{ord}\,1: \quad \hat{u}_2\frac{\partial \hat{u}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{u}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = 0 \tag{89}$$

$$\text{ord}\,\varepsilon^2 < \text{ord}\,\Delta < \text{ord}\,1: \quad \hat{u}_2\frac{\partial \hat{u}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{u}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = 0 \tag{90}$$

$$\text{ord}\,\varepsilon^2 = \text{ord}\,\Delta: \quad \hat{u}_2\frac{\partial \hat{u}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{u}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = -\frac{\partial \overline{\hat{u}_1^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u}_1'\hat{v}_1'}}{\partial y_\eta} + \frac{\partial^2 \hat{u}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{u}_2}{\partial y_\eta^2} \tag{91}$$

$$\text{ord}\,\Delta < \text{ord}\,\varepsilon^2: \quad \frac{\partial^2 \hat{u}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{u}_2}{\partial y_\eta^2} = 0 \tag{92}$$

y-momentum equation:

$$\text{ord}\,\Delta = \text{ord}\,1: \quad \hat{u}_2\frac{\partial \hat{v}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{v}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial y_\eta} = 0 \tag{93}$$

$$\text{ord}\,\varepsilon^2 < \text{ord}\,1 < \text{ord}\,\Delta: \quad \hat{u}_2\frac{\partial \hat{v}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{v}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial y_\eta} = 0 \tag{94}$$

$$\text{ord}\,\varepsilon^2 = \text{ord}\,\Delta: \quad \hat{u}_2\frac{\partial \hat{v}_2}{\partial x_\Delta} + \hat{v}_2\frac{\partial \hat{v}_2}{\partial y_\eta} + \frac{\partial \hat{p}_2}{\partial x_\Delta} = \frac{\partial \overline{\hat{v}_1'^2}}{\partial x_\Delta} - \frac{\partial \overline{\hat{u}_1'\hat{v}_1'}}{\partial y_\eta} + \frac{\partial^2 \hat{v}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{v}_2}{\partial y_\eta^2} \tag{95}$$

$$\text{ord}\,\Delta < \text{ord}\,\varepsilon^2: \quad \frac{\partial^2 \hat{v}_2}{\partial x_\Delta^2} + \frac{\partial^2 \hat{v}_2}{\partial y_\eta^2} = 0 \tag{96}$$

Note that in region $(\Delta, \eta) = (\varepsilon^2, \varepsilon^2)$ the full Navier-Stokes averaged equation is recovered. The leading order equations for $\hat{u}_1$ together with the no-slip condition at the wall gives $\hat{u}_1 = 0$.

In Cruz and Silva Freire (1998) the asymptotic structure of the thermal turbulent boundary layer is also studied through Kaplun limits. The procedure is basically the same and the results similar to those derived for the velocity field. For more details concerning this problem the reader is referred to the original work. There new formulations are advanced for the law of the wall for the velocity and the temperature fields for separating flows. To these formulations, experimental and numerical validation are given based on the works of Vogel and Eaton (1985) and Driver and Seegmiller (1995).

The resulting asymptotic structure for both the velocity and the temperature boundary layers is shown in Fig. 2, which was taken from Cruz and Silva Freire (1998). This figure incorporates the dependence of the structure on the Prandtl number.

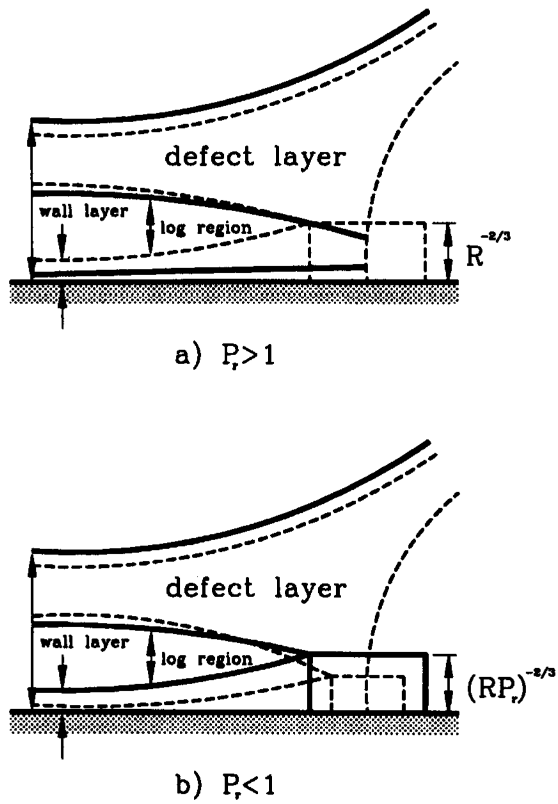

Fig. 2 Flow asymptotic structure for the separation problem.

The Shock-Interacting Boundary Layer

For a compressible flow, the two-dimensional Navier-Stokes equations of mean motion can be cast in terms of a mass-weighted-averaging procedure. The continuity and the momentum equations can then be written in the following nondimensional form:

$$\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_j\right) = 0 \tag{97}$$

$$\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_i\tilde{u}_j\right) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(-\overline{\rho u_i' u_j'} + \frac{1}{R}\tau_{ij}\right) \tag{98}$$

where the stress tensor τ_{ij} is given by

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial \tilde{u}_l}{\partial x_l} + \mu\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) \tag{99}$$

These equations are complemented by the energy and the state equations. It follows that

$$a^2 = \frac{1}{2}(\gamma + 1) - \frac{1}{2}(\gamma - 1)\tilde{u}_i\tilde{u}_j, \qquad a = \sqrt{\frac{\bar{p}}{\bar{\rho}}} \tag{100}$$

$$\bar{p} = \bar{\rho}\bar{t} \tag{101}$$

In the above equations, x, u, p, t, and ρ have their classic meaning. λ is the bulk viscosity ($= -2/3\mu$), μ is the viscosity and δ_{ij} the Kronecker delta. The nondimensional velocities, pressure, temperature, and density are all referred to their critical values just outside the boundary layer and ahead of the shock. R denotes the Reynolds number. The dashes denote turbulent fluctuations; the bars and the tildes denote, respectively, conventional time averaging and mass-weighted averaging. In what follows, for the sake of simplicity, the bars and the tildes will be omitted.

The order of magnitude of the turbulent terms in the equations of motion can be estimated through the experimental results of Kistler (1959), Kistler and Chen (1963), and Morkovin (1962). These authors have shown that: (a) u', ρ', and t' have the same order of magnitude and (b) the root square-mean value of p' is proportional to u'. Thus, in view of the above remarks, the scales of fluctuation can be written as

$$\text{ord}(u') = \text{ord}(v') = \text{ord}(\rho') = \text{ord}(t') = \text{ord}(u_\tau) \tag{102}$$

$$\text{ord}(p') = \text{ord}(u_\tau^2) \tag{103}$$

To find the asymptotic structure of the flow in the interaction region, we consider the same stretching transformation used in the two previous sections of this article, that is,

$$x_\Delta = \frac{x}{\Delta(\varepsilon)}, \quad y_\eta = \frac{y}{\Delta(\varepsilon)}, \quad \hat{u}_i(x_\Delta, y_\Delta) = u_i(x, y) \tag{104}$$

with $\Delta(\varepsilon)$ and $\eta(\varepsilon)$ defined on Ξ.

Following previous studies of the problem by Messiter (1980) and by Silva Fieire (1988), in the interaction region we separate the asymptotic expansions for the solution into a rotational and an irrotational part.

Thus, we introduce here the two small parameters

$$\varepsilon = \frac{U_\infty}{a^*} - 1 \tag{105}$$

and

$$u_\tau = \frac{1}{a^*}\sqrt{\frac{\tau_w}{\rho_w}} \tag{106}$$

where a^* is the critical sound speed in the external flow just ahead of the shock, τ_w is the laminar stress at the wall, and ρ_w is the density at the wall. From now on, the reader is asked not to confuse the new definition of ε with its previous definition in the previous sections.

As a result, the velocity profiles can be written as

$$u = 1 + \varepsilon u_\alpha(x, y) + u_\tau u_\beta(y) \tag{107}$$

$$u = \varepsilon^{3/2} v_\alpha(x, y) \tag{108}$$

where u_α and u_β represent, respectively, the irrotational and the rotational parts of the flow.

After substitution of Eqs. (107) and (108) into the equations of motion, and passage of the η-limit process onto the resulting equations, we get for the x-momentum equation:

$$\text{ord } \eta = 1: \quad \frac{\partial}{\partial x_\Delta}\left(\rho \hat{u}_\alpha \hat{u}_\alpha\right) + \frac{\partial}{\partial y_\eta}\left(\rho \hat{u}_\alpha \hat{v}_\alpha\right) = -\frac{\partial \hat{p}_\alpha}{\partial x_\Delta} \tag{109}$$

$$\text{ord } u_\tau^2 < \text{ord } \eta < 1: \quad \frac{\partial}{\partial x_\Delta}\left(\rho \hat{u}_\alpha \hat{u}_\alpha\right) + \frac{\partial}{\partial y_\eta}\left(\rho \hat{u}_\alpha \hat{v}_\alpha\right) = -\frac{\partial \hat{p}_\alpha}{\partial x_\Delta} \tag{110}$$

$$\text{ord } \eta = \text{ord } u_\tau^2: \quad \frac{\partial}{\partial x_\Delta}\left(\rho \hat{u}_\alpha \hat{u}_\alpha\right) + \frac{\partial}{\partial y_\eta}\left(\rho \hat{u}_\alpha \hat{v}_\alpha\right) = -\frac{\partial \hat{p}_\alpha}{\partial x_\Delta} + \frac{\partial}{\partial y_\eta}\left(-\rho \overline{\hat{u}'_\alpha \hat{v}'_\alpha}\right) \tag{111}$$

$$\text{ord } 1/Ru_\tau < \text{ord } \eta < \text{ord } u_\tau^2: \quad \frac{\partial}{\partial y_\eta}\left(-\rho \overline{\hat{u}'_\alpha \hat{v}'_\alpha}\right) = 0 \tag{112}$$

$$\text{ord } \eta = \text{ord } 1/Ru_\tau: \quad \frac{\partial}{\partial y_\eta}\left(-\rho \overline{\hat{u}'_\alpha \hat{v}'_\alpha}\right) + \mu \frac{\partial^2 \hat{u}_\beta}{\partial y_\eta^2} = 0 \tag{113}$$

$$\text{ord } \eta < \text{ord } 1/Ru_\tau: \quad \mu \frac{\partial^2 \hat{u}_\beta}{\partial y_\eta^2} = 0 \tag{114}$$

Because we are considering the flow in the interaction region, in passing the η-limit we have taken ord(Δ) = ord(ε). The other equations, continuity, energy, and state, do not give any contribution to the asymptotic structure. In fact, Silva Freire (1989) has shown that if the full energy equation is considered, and the concepts of section two are applied to the full set of equations, then the overlap domains of the velocity field and of the temperature field will coincide.

The continuity equation simply implies that

$$\text{ord } (v) < \frac{\eta}{\Delta} \text{ord } (u) \tag{115}$$

The classic two-deck structure of the turbulent boundary layer is then clearly seen from Eqs. (109) to (114). Note that Eqs. (111) and (113) are the principal equations; their overlap domain is identical to the overlap domain determined for the incompressible flow case.

In the vicinity of the shock wave, however, the asymptotic structure above deduced must change. The strong pressure gradient imparted to the boundary layer by the shock wave alters the balance of terms in the equations of motion, giving rise to a new structure where for most of the boundary layer the problem becomes an inviscid one. The need for the establishment of an inviscid rotational flow model for the description of the interaction has been recognized because Lighthill (1953) proposed his linearized solution for the laminar problem. The result is that all recent theories advanced for the turbulent problem must somehow accommodate the inviscid rotational interaction model without contradicting the features of the turbulent flow. To surmount this difficulty, the theories of Melnik and Grossmann (1974), Adamson and Feo (1975), Messiter (1980), and Liou and Adamson (1980) consider the introduction of a blending region in the interaction region. The blending layer is, in fact, nothing more than the turbulent region defined by the overlap domain and derived by our asymptotic analysis of the problem. The absence of an equation similar to Eq. (112) in the matched asymptotic expansions method is the main reason for the difficulties this method presents. Likewise, this is the reason why the structure depicted by Eqs. (111) to (114) can deal with the interaction problem.

To take into account the presence of the shock wave, we pass the Δ-limit process onto Eqs. (109) through (114). The result is

$$\text{ord }\Delta = \text{ord }\varepsilon: \quad \frac{\partial}{\partial x_\Delta}(\rho\hat{u}_\alpha\hat{u}_\alpha) + \frac{\partial}{\partial y_\eta}(\rho\hat{u}_\alpha\hat{v}_\alpha) = -\frac{\partial\hat{p}_\alpha}{\partial x_\Delta} \tag{116}$$

$$\text{ord }\varepsilon/Ru_\tau^3 < \text{ord }\Delta < \text{ord }\varepsilon\eta/u_\tau^2: \quad \frac{\partial}{\partial x_\Delta}(\rho\hat{u}_\alpha\hat{u}_\alpha) + \frac{\partial}{\partial y_\eta}(\rho\hat{u}_\alpha\hat{v}_\alpha) = -\frac{\partial\hat{p}_\alpha}{\partial x_\Delta} \tag{117}$$

$$\text{ord }\Delta = \text{ord }\varepsilon/Ru_\tau^3: \quad \frac{\partial}{\partial x_\Delta}(\rho\hat{u}_\alpha\hat{u}_\alpha) + \frac{\partial}{\partial y_\eta}(\rho\hat{u}_\alpha\hat{v}_\alpha) = -\frac{\partial\hat{p}_\alpha}{\partial x_\Delta} + \frac{\partial}{\partial y_\eta}\left(-\rho\overline{\hat{u}'_\alpha\hat{v}'_\alpha}\right) + \mu\frac{\partial^2\hat{u}_\beta}{\partial y_\eta^2} \tag{118}$$

$$\text{ord }\Delta < \text{ord }\varepsilon/Ru_\tau^3: \quad \mu\frac{\partial^2\hat{u}_\beta}{\partial y_\eta^2} = 0 \tag{119}$$

The change in the asymptotic structure of the flow in the interaction region is noticeable from the above equations. In particular, we note that as the shock is approached, that is, as the order of magnitude of Δ increases, the validity domain of the outer principal equation changes position until the two principal equations merge at $(\Delta, \eta) = (\varepsilon/(u_\tau^3R), 1/(u_\tau R))$. Indeed, as shown by the calculations, at the beginning of the interaction the outer principal equation is positioned at $(\Delta, \eta) = (\varepsilon, u_\tau^2)$. However, as the order of magnitude of η varies from u_τ^2 to $1/u_\tau R$, this equation moves along the path $(\varepsilon\eta/u_\tau^2, \eta)$ until reaching the point $(\varepsilon / u_\tau^3 R, 1/u_\tau R)$. The flow structure is then shown to reduce from a classic two-deck structure to a one-deck structure near to the foot of the shock wave. According to these results, there is a region at the foot of the shock where the full boundary layer equations are recovered.

The results of this section are shown in Fig. 3; they will be compared with the experimental data of Sawyer and Long (1982).

Figures 4 and 5 reproduce, from the experimental data, a map that indicates the dominant region of every term in the equations of motion. In both cases, Mach numbers of 1.27 and of 1.37 are shown. The meaning of the shades in gray is clear. Thus, the furthest from the wall tone corresponds to the inertia and pressure gradient terms, the intermediate tone to the turbulent terms, and the remaining tone to the viscous terms. The shock wave is located at $x = 0$. Observe, as predicted by the asymptotic theory, the complete dominance of the inertia and pressure terms in the vicinity of the shock. This feature is particularly striking for the 1.37 Mach number case where the influence of the shock extends down to the viscous layer. The consequence is that the phenomenon is, for most of the interaction region, and, to a leading order, governed by inviscid equations.

FINAL REMARKS

In the first part of this article, some ideas of Kaplun concerning limit processes have been extended to higher orders through the proposition of the principal equations. This result is central to our work, because it ensures that the asymptotic structure of a singular perturbation problem can be determined uniquely by a first-order analysis of the formal domains of validity. The resulting principal equations are expected to be satisfied by the corresponding limits of the exact solution, providing approximate solutions that overlap and cover the entire domain of validity.

In the second part of this article, application of Kaplun limits to the equations of motion has shown the zero-pressure turbulent boundary layer to have a two-deck structure,

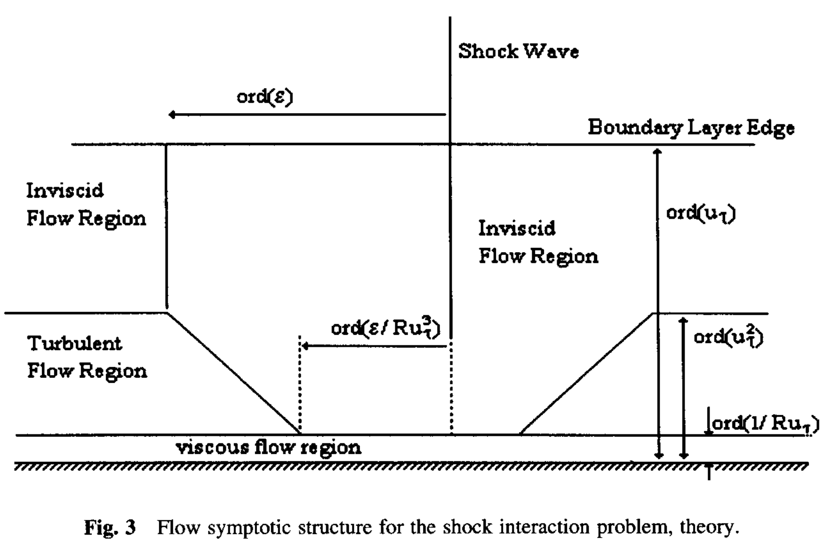

Fig. 3 Flow symptotic structure for the shock interaction problem, theory.

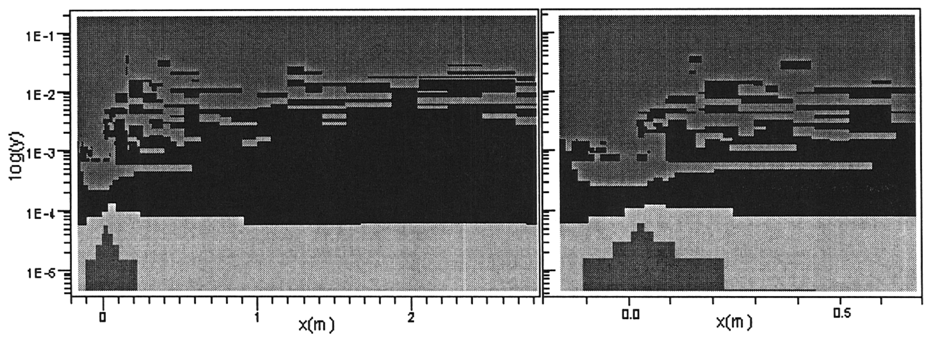

Fig. 4 Flow asymptotic structure for the shock interaction problem. Data from Sawyer and Long (1982). Mach number = 1.27.

the principal equations being located at points (ε^2, 1) and ($1/\varepsilon R$, 1) of the product space ($\Xi \times \Sigma$). The present results are very much in accordance with the earlier works of Yajnik, Mellor, and Bush and Fendell. They seem to corroborate the idea that a one-parameter theory can correctly describe the flow structure and, furthermore, do not give any evidence to suggest the contrary.

The present analysis has also shown how the two-deck turbulent boundary layer structure develops into a one-deck structure near a separation point. This results seems, at first, contradictory to the three-layer structure found by other authors (Melnik [1989], Durbin and Belcher [1992]). However, we point out that all local equations derived by these authors are intermediate equations, in the sense of Kaplun, therefore being contained in the domain of validity of the principal equations derived here. In other words, we may say that those theories are "contained" in the present theory. Of course, the principal equations are of difficult solution, do not provide closed analytical solutions. However, only these equations provide fundamental insight into how the viscous and defect layers merge as a separation point is approached.

Concerning the problem of interaction between a shock-wave and a turbulent boundary layer, the application of Kaplun limits to the equations of motion has shown the flow

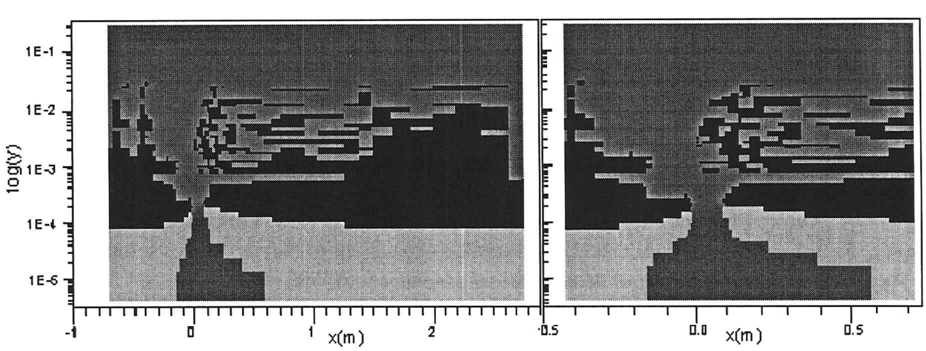

Fig. 5 Flow asymptotic structure for the shock interaction problem. Data from Sawyer and Long (1982). Mach number = 1.37.

to attain a one-deck structure, which is distinct from those of other authors, but consistent with the experimental data of Sawyer and Long (1982), and with the general knowledge of the problem we have. The theory, as presented here, can formally explain how an inviscid flow region is formed at the foot of the shock wave, resulting from the disappearance of the fully turbulent region.

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