THE UNIFIED INTEGRAL TRANSFORMS (UNIT) ALGORITHM WITH TOTAL AND PARTIAL TRANSFORMATION

Renato M. Cotta,[1,2,*] Diego C. Knupp,[3] Carolina P. Naveira-Cotta,[2] Leandro A. Sphaier,[4] & João N. N. Quaresma[5]

[1]*Laboratory of Transmission and Technology of Heat, LTTC, Department of Mechanical Engineering, Universidade Federal do Rio de Janeiro, POLI&COPPE/UFRJ, Cidade Universitária, Cx. Postal 68503, 21945-970, Rio de Janeiro, RJ, Brazil*

[2]*Laboratory of Microfuidics and Micro-systems, LabMEMS, Department of Mechanical Engineering, Universidade Federal do Rio de Janeiro, POLI&COPPE/UFRJ, Cidade Universitária, Cx. Postal 68503, 21945-970, Rio de Janeiro, RJ, Brazil*

[3]*Laboratory of Experimentation and Numerical Simulation in Heat and Mass Transfer, LEMA, Dept. of Mechanical Engineering and Energy, Instituto Politécnico, IPRJ/UERJ, Rua Bonfim, 25, Nova Friburgo, RJ, 28624-570, Brazil*

[4]*Laboratory of Theoretical and Applied Mechanics, LMTA, Mechanical Engineering Department, Universidade Federal Fluminense, PGMEC/UFF, Rua Passo da Pátria 156, Bloco E, Sala 216, 24210-240, Niterói, RJ, Brazil*

[5]*School of Chemical Engineering, Universidade Federal do Pará, FEQ/UFPA, Campus Universitário do Guamá, 66075-110, Belém, PA, Brazil*

[*]Address all correspondence to Renato M. Cotta E-mail: cotta@mecanica.coppe.ufrj.br

The theory and algorithm behind the open-source mixed symbolic-numerical computational code named UNIT (unified integral transforms) are described. The UNIT code provides a computational environment for finding solutions of linear and nonlinear partial differential systems via integral transforms. The algorithm is based on the well-established analytical-numerical methodology known as the generalized integral transform technique (GITT), together with the mixed symbolic-numerical computational environment provided by the Mathematica system (version 7.0 and up). This paper is aimed at presenting a partial transformation scheme option in the solution of transient convective-diffusive problems, which allows the user to choose a space variable not to be integral transformed. This approach is shown to be useful in situations when one chooses to perform the integral transformation on those coordinates with predominant diffusion effects only, whereas the direction with predominant convection effects is handled numerically, together with the time variable, in the resulting transformed system of one-dimensional partial differential equations. Test cases are selected based on the nonlinear three-dimensional Burgers' equation, with the establishment of reference results for specific numerical values of the governing parameters. Then the algorithm is illustrated in the solution of conjugated heat transfer in microchannels.

KEY WORDS: *integral transforms, hybrid methods, symbolic computation, Burgers' equation, conjugated heat transfer, microchannel flow*

NOMENCLATURE

D_h	hydraulic diameter in the conjugated heat transfer problem
d	dissipation operator coefficient
g	nonlinear source term
K	diffusion operator coefficient
L_e	distance from the channel centerline to the external face of the channel wall in the conjugated heat transfer problem
L_f	channel height in the conjugated heat transfer problem
L_w	channel width in the conjugated heat transfer problem
M	number of subregions in semianalytical and Gaussian integration
N	truncation order in eigenfunction expansion
n	number of coupled potentials in the general problem formulation
Pe	Péclet number in the conjugated heat transfer problem
T	dimensionless potential in the general problem formulation; potential in Burgers' equations (9) and (10); temperature field in the conjugated
t	dimensionless time variable heat transfer problem, Eqs. (11)
U	dimensionless fully developed velocity profile in the conjugated heat transfer problem
u	nonlinear function in convection term in Burgers' equation (9); fully developed velocity profile in the conjugated heat transfer problem, Eqs. (11)
u_0	linear parameter in nonlinear convection term in Burgers' equation, Eqs. (9)
w	transient operator coefficient in general problem; heat capacity in the conjugated heat transfer problem, Eqs. (11)
$\mathbf{x}$	position vector in the general problem formulation
x	longitudinal coordinate in problem (9)
Y	dimensionless transversal coordinate in the conjugated heat transfer problem
y	transversal coordinate in problems (9) and (11)
Z	dimensionless longitudinal coordinate in the conjugated heat transfer problem
z	transversal coordinate in problem (9); longitudinal coordinate in problem (11)

Greek Symbols

α	boundary condition coefficient
β	boundary condition coefficient
θ	dimensionless temperature field in the conjugated heat transfer problem
μ	eigenvalues
ν	diffusion coefficient in Burgers' equation
ϕ_k	source term in boundary condition
ψ	eigenfunctions

Subscripts & Superscripts

i	order of eigenquantities
k	quantity corresponding to the equation of the kth potential in general problem
-	position vector excluding the space variable not to be integral transformed
–	integral transform
$\sim$	normalized eigenfunction

1. INTRODUCTION

Integral transforms have been successfully used in different branches of the physical, mathematical, and engineering sciences for about 200 years. Its introduction can be attributed to Fourier, after the publication of his treatise on the analytical theory of heat (Fourier, 1822). In essence, Fourier at that time advanced the idea of separation of variables, so as to handle and interpret the solutions of the newly derived heat conduction equation, after proposing the constitutive equation known nowadays as Fourier's law. He gave a series of examples before stating that an arbitrary function defined on a finite interval can be expanded in terms of a trigonometric series that is now known as the Fourier series. In an attempt to extend his new ideas to functions defined on an infinite interval, Fourier discovered an integral transform and its inversion formula which are now commonly known as the

Fourier transform and the inverse Fourier transform, respectively. His work provided the modern mathematical theory of heat conduction, but also introduced Fourier series, Fourier integrals, and stated an important result that is known as the Fourier integral theorem, later rephrased by Dirichlet (Debnath and Bhatta, 2007).

The method of integral transforms was then widely used in solving linear partial differential equations (PDEs) of mathematical physics along the following years, and the classical treatise of Carslaw and Jaeger (1947) provides a wide collection of solutions obtained in heat conduction theory by this and the competing analytical approaches then available. According to Luikov (1980), father of modern analytical heat diffusion theory, as detailed in his most classical work (Luikov, 1968), it was not until the work of Koshlyakov (1936) that an idea was provided in handling nonhomogeneous diffusion equations and boundary conditions by the method of finite integral transformations, and the theory of such integral transforms was developed in detail by Grinberg (1948), who also extended this approach to multilayer problems, considering step variations of the material properties along the transformation coordinate. A very active period of research on exact analytical solutions of nonhomogeneous heat and mass transfer problems then followed, when besides the continuous contribution of Luikov and coworkers, one should certainly include the contributions of other very prominent researchers, such as Olçer (1964, 1967), Ozisik (1968), and Mikhailov (1967, 1972). This period was so fruitful for analytical heat and mass transfer that Luikov himself, in 1974, contacted both Ozisik and Mikhailov regarding the joint publication of books on heat conduction and convection. However, Luikov passed away even before the startup of these projects, but Mikhailov and Ozisik finally met each other in 1976 and started a new project inspired by Luikov's suggestions, work that would be completed and published only in 1984, when most of the available exact solutions of heat and mass diffusion through integral transforms were unified in seven different classes of problems and systematically presented in a reference book (Mikhailov and Ozisik, 1984). This book was very much influenced by the previous publications of Mikhailov along a very productive period of more than one decade (including Mikhailov, 1967, 1972, 1973a,b, 1975, 1977a,b; Mikhailov and Shishedjiev, 1976; Mikhailov and Ozisik, 1980, 1981; Mikhailov et al., 1982; Mikhailov and Vulchanov, 1983), when he challenged the integral transform method to handle different classes of unified formulations in heat and mass diffusion. Ozisik also pursued the rewriting of his 1968 textbook during this period (Ozisik, 1968) and prepared a fairly complete new work on heat conduction, compiling different analytical, approximate, and numerical approaches (Ozisik, 1980).

Within this same period, Ozisik and Mikhailov, independently, also perceived the limitations of the integral transforms method, as it was known at that time, when they tried to solve problems with time-dependent boundary condition coefficients (Ozisik and Murray, 1974; Yener and Ozisik, 1974) or time-dependent equation coefficients (Mikhailov, 1975). In these early works, the integral transformation process, due to the time dependence of the transformation kernel represented by the eigenvalues, eigenfunctions, and norms, was not successful in fully transforming the original PDE and resulted in a coupled infinite ordinary differential system for the transformed temperature fields. Nevertheless, the authors were still able to propose analytical approximations by taking only a limited number of terms in the coupling terms and then forcing the simplification to a decoupled system. In the work of Ozisik and Murray (1974), the expression generalized integral transform technique was employed for the first time. This same concept of approximate analytical solution was later on employed by Ozisik and Guçeri (1977) in the solution of phase change problems, by Bayazitoglu and Ozisik (1980) in the analysis of internal forced convection with axial diffusion effects, by Bogado Leite et al. (1980, 1982) in solving moving boundary diffusion problems related to the erosion of fusion reactor walls, and by Cotta and Ozisik (1985) in the solution of transient internal convective heat transfer due to wall temperature variations. Such approximate solutions, though elegant and easy to compute, had limitations in terms of accuracy, within certain ranges of the involved parameters and independent variables, and would also require some sort of numerical solution for verification purposes. However, in Cotta (1986a) the complete solution of the coupled transformed system was achieved, based on the numerical solution of a truncated version of the transformed ordinary differential equation (ODE' system, as obtained for a diffusion problem with a prescribed moving boundary, associated with oxidation of nuclear fuel rods cladding. The resulting transformed system typical of such integral transformations is likely to present significant stiffness, especially for larger truncation orders, but at that time reliable solvers for stiff initial value problems were already available, allowing for error-controlled solutions of the transformed potentials. Only then would the GITT (generalized integral transform technique) be proposed as a full hybrid numerical-analytical solution of nontransformable diffusion or convection–diffusion problems. The GITT was later advanced to offer analytical solutions of the complete transformed system for linear problems, such as in the transient internal convection problem in the complex domain solved in Cotta and Ozisik (1986). In a natural sequence, the previous solutions of diffusion problems with variable equation or boundary coefficients were formalized soon afterward, under the new concept of obtaining the solution of the complete transformed systems (Cotta, 1986b; Cotta and Ozisik, 1987). Clearly, the generalized approach was then interpreted as the closest in nature to the exact solutions obtainable by integral transforms in the case of transformable problems, although somehow still approximate due to the truncation of the infinite transformed system. Once the transformed system is numerically solved by controlling the relative error in the initial value problem algorithm, one is left with the task of controlling the global error of the solution by adequately choosing the system size and thus the eigenfunction expansion truncation order.

An avenue of opportunities was then opened and the successive challenges for the GITT extension would form the basis of a series of theses and papers. Just to name a few contributions within this fruitful period, the analysis of diffusion within irregular domains was soon proposed (Aparecido and Cotta, 1987; Aparecido et al., 1989), and followed by the analysis of nonlinear diffusion problems (Cotta, 1990; Serfaty and Cotta, 1990, 1992), conjugated convection-conduction problems (Guedes et al., 1989; Guedes and Cotta, 1991), ablation moving boundary problems (Diniz et al., 1990), boundary layer equations (Cotta and Carvalho, 1991; Carvalho et al., 1993), Navier-Stokes equations (Perez Guerrero and Cotta, 1992), drying problems (Ribeiro et al., 1993), and natural convection in porous media (Baohua and Cotta, 1993). Based on the above works and a few others, the first reference and textbook on the GITT was prepared and published in 1993 (Cotta, 1993), including some formal aspects of the approach that were not previously dealt with in the available publications. This effort made the hybrid approach more visible and the positive response of the heat transfer community was soon provided, as demonstrated in the keynote lecture at the 10th IHTC, UK, in 1994 (Cotta, 1994a) and the invited review paper in the *International Journal of Heat and Mass Transfer* in that same year (Cotta, 1994b), celebrating the contribution of Prof. James P. Hartnett to the heat and mass transfer field. Two years later a book on heat conduction was published (Cotta and Mikhailov, 1997) that would combine the knowledge on improved lumped system analysis, GITT, and symbolic computation. Also within this phase, a compilation of advanced contributions was organized (Cotta, 1998) in order to complement the 1993 book, which remains an important source of advanced reference work on the generalized integral transform technique.

The next phase then initiated in the development of integral transforms in heat and fluid flow was characterized by the optimization of the numerical tasks, proposition of more challenging problems among those classes already handled, and the more ample application of this knowledge basis in different areas. For instance, one may recall the analysis of fluid flow and mass transfer within petroleum reservoirs (Almeida and Cotta, 1995), three-dimensional Navier-Stokes equations (Quaresma and Cotta, 1997), Navier-Stokes equations in irregular geometries (Perez Guerrero et al., 2000), forced and natural convection with variable fluid properties (Machado and Cotta, 1999; Leal et al., 2000), compressible flow and heat transfer in ultracentrifuges (Pereira et al., 2002), stability analysis in natural convection (Alves et al., 2002), three-dimensional natural convection in porous enclosures (Luz Neto et al., 2002), eigenvalue and diffusion problems in multidimensional irregular domains (Sphaier and Cotta, 2000; Sphaier and Cotta, 2002), and contaminant dispersion in fractured media (Cotta et al., 2003), to name a few. The maturity of the approach was then consolidated in the edited book (Santos et al., 2001), in the invited editorial of the *Heat Transfer Engineering* journal (Cotta and Orlande, 2003), and finally in the invited chapter for the *Handbook of Numerical Heat Transfer* (Cotta and Mikhailov, 2006).

In recent years, besides the continuous search for more challenging problems and different application areas, emphasis has been placed in unifying and simplifying the use of the GITT, to reach a larger number of users and offer an alternative hybrid solution to their problems. Hybrid methods become even more powerful and applicable when symbolic manipulation systems, which were also widely disseminated along the last two to three decades, are employed. The effort in integrating the knowledge on the GITT into a symbolic-numerical algorithm resulted in the so-called UNIT code (unified integral transforms), initially proposed as a project in 2007 and intended to bridge the gap between simple problems that allow for a straightforward analytical solution, and more complex and involved situations that almost unavoidably require specialized software systems. The open-source UNIT code is thus an implementation and development platform for researchers and engineers interested in integral transform solutions of diffusion and convection–diffusion problems (Sphaier et al., 2011; Knupp et al., 2010, Cotta et al., 2010, 2013).

This work is thus aimed at reviewing the current version of the multidimensional UNIT code in the Mathematica platform and introduces its partial transformation scheme option. In this alternative partial transformation mode, the user is allowed to choose one of the dimensional variables not to be integral transformed. This strategy can be very useful for the solution of transient convective–diffusive problems in which one chooses to perform the integral transformation only on the directions with predominant diffusion effects, whereas along the direction with predominant convection effects the problem is solved numerically, together with the time variable, in the resulting transformed system of one-dimensional PDEs. In order to illustrate the present implementation, we show some results regarding a nonlinear three-dimensional formulation of Burgers' equation, which are critically compared to the default total transformation UNIT code solutions. Also, we illustrate the partial transformation approach for the analysis of conjugated heat transfer in microheat spreaders made of microchannels molded in a polymeric matrix (Ayres et al., 2011), employing a recently advanced single-domain strategy for conjugated heat transfer problems developed by Knupp et al. (2012, 2013a,b, 2014a,b).

2. PROBLEM FORMULATION

A general transient convection–diffusion problem of n coupled potentials is considered, defined in the region V with boundary surface S:

$$w_k(\mathbf{x})L_{k,t}T_k(\mathbf{x},t) = G_k(\mathbf{x},t,\mathbf{T}), \quad \mathbf{x}\in V, \quad t>0, \quad k=1,2,...,n \tag{1a}$$

where the t operator $L_{k,t}$ for a parabolic or parabolic–hyperbolic formulation is given by

$$L_{k,t} \equiv \frac{\partial}{\partial t} \tag{1b}$$

and

$$G_k(\mathbf{x},t,\mathbf{T}) = \nabla\cdot(K_k(\mathbf{x})\nabla T_k(\mathbf{x},t)) - d_k(\mathbf{x})T_k(\mathbf{x},t) + g_k(\mathbf{x},t,\mathbf{T}) \tag{1c}$$

with initial and boundary conditions given, respectively, by

$$T_k(\mathbf{x},0) = f_k(\mathbf{x}),\ \mathbf{x}\in V \tag{1d}$$

$$\left[\alpha_k(\mathbf{x}) + \beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial n}\right]T_k(\mathbf{x},t) = \varphi_k(\mathbf{x},t,\mathbf{T}), \quad \mathbf{x}\in S, \quad t>0 \tag{1e}$$

where **n** denotes the outward-drawn normal to the surface S, and where the potentials vector is given by

$$\mathbf{T} = \{T_1, T_2, ..., T_k, ..., T_n\} \tag{1f}$$

As mentioned before, Eqs. (1a)–(1f) are quite general, since nonlinear and convection terms may be grouped into the equations and boundary conditions source terms. It may be highlighted that in the case of decoupled linear source terms, i.e., $g \equiv g(\mathbf{x},t)$, and $\varphi \equiv \varphi(\mathbf{x},t)$, this example is reduced to a class I linear diffusion problem for each potential, according to the classification in Mikhailov and Ozisik (1984), and exact analytical solutions are readily available via the classical integral transform technique. Otherwise, this problem shall not be *a priori* transformable, except for a few linear coupled situations also illustrated in Mikhailov and Ozisik (1984). However, the formal solution procedure provided by the GITT (Cotta, 1993) may be invoked in order to provide hybrid numerical-analytical solutions for the nontransformable problems.

The formal solution regarding the standard procedure of the UNIT code is known as the total transformation scheme, described in Cotta et al. (2013), in which all spatial variables are integral transformed. Here we focus on the partial integral transformation scheme option of the UNIT code, as an alternative solution path to problems with a strong convective direction, which is not eliminated through integral transformation but kept within the transformed system.

3. SOLUTION METHODOLOGY

3.1 Total Transformation

Following the formal solution procedure for nonlinear convection–diffusion problems through integral transforms, the proposition of eigenfunction expansions for the associated potentials are first required. The preferred eigenvalue problem choice appears from the direct application of the separation of variables methodology to the linear homogeneous purely diffusive version of the proposed problem. Thus the recommended set of decoupled auxiliary problems is here given by

$$\nabla\cdot[K_k(\mathbf{x})\nabla\psi_{ki}(\mathbf{x})] + [\mu_{ki}^2 w_k(\mathbf{x}) - d_k(\mathbf{x})]\psi_{ki}(\mathbf{x}) = 0, \quad \mathbf{x}\in V \tag{2a}$$

$$\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]\psi_{ki}(\mathbf{x})=0, \quad \mathbf{x}\in S \tag{2b}$$

where the eigenvalues μ_{ki} and associated eigenfunctions $\psi_{ki}(\mathbf{x})$ are assumed to be known from exact analytical expressions, for instance, obtained through symbolic computation (Wolfram, 2005) or application of the GITT itself (Naveira-Cotta et al., 2009). One should notice that Eqs. (1a)–(1f) are presented in a form that already reflects this choice of eigenvalue problems, given by Eqs. (2a) and (2b), with the adoption of linear coefficients in both the equations and boundary conditions, and incorporating the remaining terms (coupling, nonlinear, and convective terms) into the general nonlinear source terms, without loss of generality.

By making use of the orthogonality properties of the eigenfunctions, it is then possible to define the following integral transform pairs:

$$\bar{T}_{ki}(t)=\int_V w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x})T_k(\mathbf{x},t)dV \quad \text{transforms} \tag{3a}$$

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}\tilde{\psi}_{ki}(\mathbf{x})\bar{T}_{k,i}(t) \quad \text{inverses} \tag{3b}$$

where the symmetric kernels $\tilde{\psi}_{ki}(\mathbf{x})$ are given by

$$\tilde{\psi}_{ki}(\mathbf{x})=\frac{\psi_{ki}(\mathbf{x})}{\sqrt{N_{ki}}}; \quad N_{ki}=\int_V w_k(\mathbf{x})\psi_{ki}^2(\mathbf{x})\mathrm{d}V \tag{3c,d}$$

with N_{ki} being the normalization integral.

The integral transformation of Eq. (1a) is accomplished by applying the operator $\int_V \tilde{\psi}_{ki}(\mathbf{x})\,(\cdot)\,\mathrm{d}V$ and making use of the boundary conditions given by Eqs. (1e) and (2b), yielding

$$\frac{d\bar{T}_{ki}(t)}{dt}+\mu_{ki}^2\bar{T}_{ki}(t)=\bar{g}_{ki}(t,\overline{\mathbf{T}})+\bar{b}_{ki}(t,\overline{\mathbf{T}}),$$
$$i=1,2,..., \quad t>0, \quad k=1,2,...,n \tag{4a}$$

where the first transformed source term $\bar{g}_{ki}(t,\overline{\mathbf{T}})$ is due to the integral transformation of the equation source terms, and the second one, $\bar{b}_{ki}(t,\overline{\mathbf{T}})$, is due to the contribution of the boundary source terms:

$$\bar{g}_{ki}(t,\overline{\mathbf{T}})=\int_V \tilde{\psi}_{ki}(\mathbf{x})g_k(\mathbf{x},t,\overline{\mathbf{T}})dV \tag{4b}$$

$$\bar{b}_{ki}(t,\overline{\mathbf{T}})=\int_S K_k(\mathbf{x})\left[\tilde{\psi}_{ki}(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial\mathbf{n}}-T_k(\mathbf{x},t)\times\frac{\partial\tilde{\psi}_{ki}(\mathbf{x})}{\partial\mathbf{n}}\right]dS \tag{4c}$$

The boundary conditions contribution may also be expressed in terms of the boundary source terms, after manipulating Eqs. (1e) and (2b), to yield

$$\bar{b}_{ki}(t,\overline{\mathbf{T}})=\int_S \varphi_k(\mathbf{x},t,\mathbf{T})\left[\frac{\tilde{\psi}_{ki}(\mathbf{x})-K_k(\mathbf{x})\frac{\partial\tilde{\psi}_{ki}(\mathbf{x})}{\partial\mathbf{n}}}{\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})}\right]dS \tag{4d}$$

The initial conditions given by Eq. (1d) are transformed through the operator $\int_V w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x})\,(\cdot)\,dV$ to provide

$$\bar{T}_{ki}(0)=\bar{f}_{ki}\equiv\int_V w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x})f_k(\mathbf{x})dV \tag{4e}$$

For the solution of the infinite coupled system of nonlinear ordinary differential equations given by Eqs. (4a)–(4e), one must make use of numerical algorithms, after the truncation of the system to a sufficiently large finite order. For instance, in the present work, the built-in routine of the Mathematica system (Wolfram, 2005) is employed, *NDSolve*, which is able to provide reliable solutions under automatic absolute and relative error control. After the transformed potentials have been numerically computed, the Mathematica routine automatically provides an interpolating function object that approximates the t variable behavior of the solution in a continuous form. Then the inversion formula can be recalled to yield the potential field representation at any desired position $\mathbf{x}$ and time t.

The solution procedure described above provides the basic straightforward working expressions for the integral transform method. Nevertheless, for an improved computational performance, it is always recommended to reduce the importance of the equation and boundary source terms so as to enhance the eigenfunction expansion's convergence behavior (Cotta and Mikhailov, 1997). The UNIT code for multidimensional applications allows for user-provided filters, but an automatic progressive linear filtering option is also implemented. The interested reader is encouraged to refer to Cotta et al. (2013), where this filtering strategy is described in detail.

The constructed multidimensional UNIT code in the Mathematica platform (Cotta et al., 2010, 2013) encompasses all of the symbolic derivations that are required in the above GITT formal solution, besides the numerical computations that are required in the solutions of the chosen eigenvalue problem and the transformed differential system. The user essentially needs to specify the problem formulation together with the required problem parameters, solve the problem using the provided UNIT algorithm, and then choose how to present the results according to the specific needs. Besides the parameters regarding the problem formulation, the user is also asked to set the truncation order of the eigenfunction expansions N and to choose the coefficients integration methodology, which can be analytical, through the Integrate routine in the Mathematica system or user-provided, semianalytical, or by an automatic Gaussian quadrature scheme that accounts for the information regarding the eigenfunctions' oscillatory behavior. The Gaussian quadratures and the alternative semianalytical integration procedure are particularly convenient in nonlinear formulations that might require costly numerical integration, once analytical integration is not feasible.

The UNIT code is here illustrated through its version 2.2.3, in Mathematica 7.0 or up, and has the following main features:

1. A system of linear or nonlinear equations (parabolic problems, parabolic–hyperbolic problems, or elliptic problems in pseudo-transient formulation);
2. Multidimensional transient formulations, automatically defined by one single parameter with the number of space dimensions;
3. Eigenvalue problem analytically solved via the DSolve routine (Sturm-Liouville problem);
4. Transformed coefficients determined by semianalytical integration (zeroth order), numerical integration (Gaussian quadrature or NIntegrate routine), or analytical integration (Integrate routine or user supplied);
5. User-defined or automatic progressive linear filtering;
6. Reordering by squared eigenvalues criterion or combination of transformed initial conditions, transformed source term, and squared eigenvalues criteria;
7. Nonhomogeneous term via Green's second formula;
8. Error estimator with adjustable residue order.

3.2 Partial Transformation

An alternative hybrid solution strategy to the above-described full integral transformation is of particular interest in the treatment of transient convection–diffusion problems with a preferential convective direction. In such cases, the partial integral transformation in all but one space coordinate may offer an interesting combination of relative advantages between the eigenfunction expansion approach and the selected numerical method for handling the coupled system of one-dimensional PDEs that results from the transformation procedure. To illustrate this partial integral transformation procedure, again a transient convection–diffusion problem of n coupled potentials is considered, but this time separating the preferential direction that is not to be integral transformed. The position vector $\mathbf{x}$ now includes the space coordinates that will be eliminated through integral transformation, here denoted by $\mathbf{x}^*$, as well as the space variable to be retained in the transformed partial differential system. Thus consider a general three-dimensional problem with $\mathbf{x} = \{x_1, x_2, x_3\}$, where only the coordinates $\mathbf{x}^* = \{x_1, x_2\}$ are intended to be eliminated by the integral transformation process, while the remaining space variable x_3 shall be retained in the transformed system to be numerically solved. The problem to be solved is now written in the following form:

$$w_k(\mathbf{x}^*)\frac{\partial T_k(\mathbf{x},t)}{\partial t} = G_k(\mathbf{x},t,\mathbf{T}), \quad \mathbf{x}\in V, \quad t>0,$$
$$k=1,2,...,n \tag{5a}$$

with

$$G_k(\mathbf{x},t,\mathbf{T}) = \nabla^*\cdot[K_k(\mathbf{x}^*)\nabla^* T_k(\mathbf{x},t)] - d_k(\mathbf{x}^*) \times T_k(\mathbf{x},t) + g_k(\mathbf{x},t,\mathbf{T}) \tag{5b}$$

where the operator ∇^* refers only to the coordinates to be integral transformed $\mathbf{x}^*$, and with initial and boundary conditions given, respectively, by

$$T_k(\mathbf{x},0) = f_k(\mathbf{x}), \quad \mathbf{x}\in V \tag{5c}$$

$$\left[\lambda_k(x_3) + \gamma_k(x_3)\frac{\partial}{\partial x_3}\right] T_k(\mathbf{x},t) = \phi_k(\mathbf{x},t,\mathbf{T}),$$
$$x_3 \in S_3, \quad t>0 \tag{5d}$$

$$\left[\alpha_k(\mathbf{x}^*) + \beta_k(\mathbf{x}^*)K_k(\mathbf{x}^*)\frac{\partial}{\partial \mathbf{n}^*}\right] T_k(\mathbf{x},t) = \varphi_k(\mathbf{x},t,\mathbf{T}),$$
$$\mathbf{x}^*\in S^*, \quad t>0 \tag{5e}$$

where $\mathbf{n}^*$ denotes the outward-drawn normal to the surface S^* formed by the coordinates $\mathbf{x}^*$ and S_3 refers to the boundary values of the coordinate x_3.

The coefficients $w_k(\mathbf{x}^*)$, $d_k(\mathbf{x}^*)$, $K_k(\mathbf{x}^*)$, $\alpha_k(\mathbf{x}^*)$, and $\beta_k(\mathbf{x}^*)$ in Eqs. (5a)–(5e) inherently carry the information on the auxiliary problem that will be considered in the eigenfunction expansion, and all the remaining terms from this rearrangement are collected into the source terms, $g_k(\mathbf{x}, t, \mathbf{T})$ and $\varphi_k(\mathbf{x}, t, \mathbf{T})$, including the existing nonlinear terms and diffusion and/or convection terms with respect to the independent variable x_3. Also, the coefficients $\lambda_k(x_3)$ and $\gamma_k(x_3)$ provide any combination of first to third kind boundary conditions in the untransformed coordinate, while the x_3 boundary source terms, $\phi_k(\mathbf{x}, t, \mathbf{T})$, collect the rearranged information that is not contained in the right-hand side of Eq. (5d).

Following the solution path previously established, the formal integral transform solution of the posed problem requires the proposition of eigenfunction expansions for the associated potentials. The recommended set of uncoupled auxiliary problems is given by

$$\nabla \cdot [K_k(\mathbf{x}^*)\nabla\psi_{ki}(\mathbf{x}^*)] + \left[\mu_{ki}^2 w_k(\mathbf{x}^*) - d_k(\mathbf{x}^*)\right] \times \psi_{ki}(\mathbf{x}^*) = 0, \quad \mathbf{x}^* \in V^* \tag{6a}$$

$$\left[\alpha_k(\mathbf{x}^*) + \beta_k(\mathbf{x}^*)K_k(\mathbf{x}^*)\frac{\partial}{\partial \mathbf{n}^*}\right]\psi_{ki}(\mathbf{x}^*) = 0, \quad \mathbf{x}^* \in S^* \tag{6b}$$

The problem indicated by Eqs. (6a) and (6b) allows, through the associated orthogonality property of the eigenfunctions, the definition of the following integral transform pairs:

$$\bar{T}_{ki}(x_3, t) = \int_{V^*} w_k(\mathbf{x}^*)\tilde{\psi}_{ki}(\mathbf{x}^*)T_k(\mathbf{x}, t)dV^*, \quad \text{transforms} \tag{7a}$$

$$T_k(\mathbf{x}, t) = \sum_{i=1}^{\infty} \tilde{\psi}_{ki}(\mathbf{x}^*)\bar{T}_{ki}(x_3, t), \quad \text{inverses} \tag{7b}$$

where the symmetric kernels $\tilde{\psi}_{ki}(\mathbf{x}^*)$ are given by

$$\tilde{\psi}_{ki}(\mathbf{x}^*) = \frac{\psi_{ki}(\mathbf{x}^*)}{\sqrt{N_{ki}}}; \qquad N_{ki} = \int_{V^*} w_k(\mathbf{x}^*)\psi_{ki}^2(\mathbf{x}^*)\mathrm{d}V^* \tag{7c,d}$$

with N_{ki} being the normalization integral.

The integral transformation of Eq. (5a) is accomplished by applying the operator $\int_{V^*} \tilde{\psi}_{ki}(\mathbf{x}^*)\,(\cdot)\,\mathrm{d}V^*$ and making use of the boundary conditions given by Eqs. (5e) and (6b), yielding

$$\frac{\partial \bar{T}_{ki}(x_3, t)}{\partial t} + \mu_{ki}^2\bar{T}_{ki}(x_3, t) = \bar{g}_{ki}(x_3, t, \overline{\mathbf{T}}) + \bar{b}_{ki}(x_3, t, \overline{\mathbf{T}}), \quad i = 1, 2, ..., \quad k = 1, 2, ..., n, \quad x_3 \in V_3, \quad t > 0, \tag{8a}$$

where the transformed source term $\bar{g}_{ki}(x_3, t, \overline{\mathbf{T}})$ is due to the integral transformation of the equation source term, and the other, $\bar{b}_{ki}(x_3, t, \overline{\mathbf{T}})$, is due to the contribution of the boundary source term at the directions being transformed:

$$\bar{g}_{ki}(x_3, t, \overline{\mathbf{T}}) = \int_{V^*} \tilde{\psi}_{ki}(\mathbf{x}^*)g_k(\mathbf{x}, t, \overline{\mathbf{T}})dV^* \tag{8b}$$

$$\bar{b}_{ki}(x_3, t, \overline{\mathbf{T}}) = \int_{S^*} K_k(\mathbf{x}^*)\left[\tilde{\psi}_{ki}(\mathbf{x}^*)\frac{\partial T_k(\mathbf{x}, t)}{\partial \mathbf{n}^*} - T_k(\mathbf{x}, t)\frac{\partial \tilde{\psi}_{ki}(\mathbf{x}^*)}{\partial \mathbf{n}^*}\right]dS^* \tag{8c}$$

The contribution of the boundary conditions at the directions being transformed may also be expressed in terms of the boundary source terms:

$$\bar{b}_{ki}(x_3, t, \overline{\mathbf{T}}) = \int_{S^*} \varphi_k(\mathbf{x}, t, \mathbf{T}) \times \left[\frac{\tilde{\psi}_{ki}(\mathbf{x}^*) - K_k(\mathbf{x}^*)\frac{\partial \tilde{\psi}_{ki}(\mathbf{x}^*)}{\partial \mathbf{n}^*}}{\alpha_k\left(\mathbf{x}^*\right) + \beta_k\left(\mathbf{x}^*\right)}\right]dS^* \tag{8d}$$

The initial conditions given by Eq. (5c) are transformed through the operator $\int_{V^*} w_k(\mathbf{x}^*)\tilde{\psi}_{ki}(\mathbf{x}^*)\,(\cdot)\,dV^*$ to provide

$$\bar{T}_{ki}(x_3, 0) = \bar{f}_{1,ki}(x_3) \equiv \int_{V^*} w_k(\mathbf{x}^*)\tilde{\psi}_{ki}(\mathbf{x}^*) \times f_k(\mathbf{x})dV^* \tag{8e}$$

Finally, the boundary conditions with respect to the direction x_3 are also transformed through the same operator, yielding

$$\left[\lambda_k(x_3) + \gamma_k(x_3)\frac{\partial}{\partial x_3}\right]\bar{T}_{ki}(x_3, t) = \overline{\phi}_{ki}(x_3, t, \overline{\mathbf{T}}), \tag{8f}$$

with

$$\overline{\phi}_{ki}(x_3, t, \overline{\mathbf{T}}) \equiv \int_{v^*} w_k(\mathbf{x}^*)\tilde{\psi}_{ki}(\mathbf{x}^*)\phi_k(\mathbf{x}, t, \mathbf{T})dV^*, \quad x_3 \in S_3, \quad t > 0 \tag{8g}$$

Equations (8a)–(8g) form an infinite coupled system of nonlinear one-dimensional PDEs for the transformed potentials $\bar{T}_{ki}(x_3, t)$, which is unlikely to be analytically solvable. Nonetheless, reliable algorithms are readily available to numerically handle this PDE system, after truncation to a sufficiently large finite order. For instance, the Mathematica system provides the built-in routine NDSolve, which can handle this system under automatic absolute and relative error control. Once the transformed potentials have been numerically computed, the

Mathematica routine automatically provides an interpolating function object that approximates the x_3 and t variables behavior of the solution in a continuous form. Then the inversion formula in Eq. (7b) can be recalled to yield the potential field representation at any desired position $\mathbf{x}$ and time t.

A major aspect in the practical implementation of this methodology is the eventual need for improving the convergence behavior of the resulting eigenfunction expansions, as pointed out in Cotta and Mikhailov (1997). The overall simplest and most effective alternative for convergence improvement appears to be the proposition of analytical filtering solutions, which present both space and time dependence within specified ranges of the numerical integration path. For instance, an appropriate quasi-steady filter for the above formulations could be written in general as

$$T_k(\mathbf{x},t) = \theta_k(\mathbf{x},t) + T_{f,k}(\mathbf{x};t) \tag{9a}$$

$$T_k(\mathbf{x}^*,x_3,t) = \theta_k(\mathbf{x}^*,x_3,t) + T_{f,k}(\mathbf{x}^*;x_3,t) \tag{9b}$$

where the second term in the right-hand sides represents the quasi-steady filter solution, which is generally sought in analytic form. The first term on the right-hand side represents the filtered potentials which are obtained through integral transformation. Once the filtering problem formulation is chosen, Eqs. (9a) and (9b) are substituted back into Eqs. (1) or (5), respectively, to obtain the resulting formulation for the filtered potential. It is desirable that the filtering solution contains as much information on the operators of the original problem as possible. This information may include, for instance, linearized versions of the source terms, so as to reduce their influence on convergence of the final eigenfunction expansions.

The analytical nature of the inversion formula allows for a direct error testing procedure at each specified position within the medium, and the truncation order N can also be controlled to fit the user global error requirements over the entire solution domain (Cotta, 1993). The tolerance testing formulas employed in the total and partial transformation are, respectively,

$$\varepsilon = \max_{\mathbf{x}\in V}\left|\frac{\sum_{i=N^*}^{N}\tilde{\psi}_{\mathrm{ki}}(\mathbf{x})\bar{T}_{k,i}(t)}{T_{f,k}(\mathbf{x};t)+\sum_{i=1}^{N}\tilde{\psi}_{\mathrm{ki}}(\mathbf{x})\bar{T}_{k,i}(t)}\right| \tag{10a}$$

$$\varepsilon = \max_{\mathbf{x}^*\in V^*}\left|\frac{\sum_{i=N^*}^{N}\tilde{\psi}_{\mathrm{ki}}(\mathbf{x}^*)\bar{T}_{k,i}(x_3,t)}{T_{f,k}(\mathbf{x}^*;x_3,t)+\sum_{i=1}^{N}\tilde{\psi}_{\mathrm{ki}}(\mathbf{x}^*)\bar{T}_{k,i}(x_3,t)}\right| \tag{10b}$$

The numerator in Eqs. (10a) and (10b) represents those terms (from orders N^* to N) that in principle might be abandoned in the evaluation of the inverse formula, without disturbing the final result to within the user-requested accuracy target, once convergence has been achieved.

4. APPLICATIONS

The UNIT partial transformation algorithm is here illustrated first with a test case based on the nonlinear three-dimensional formulation of Burgers' equation, with homogeneous and nonhomogeneous boundary conditions, previously considered by employing the total transformation procedure (Cotta et al., 2013). Then we examine the behavior of the UNIT PT algorithm for an application involving conjugated heat transfer in microchannels, as recently proposed via a single-domain formulation that encompasses both the fluid and solid regions (Knupp et al., 2012).

4.1 Nonlinear Burgers' Equation

The mathematical formulation of the three-dimensional nonlinear Burgers' equation with homogeneous boundary conditions here considered is given by

$$\frac{\partial T(x,y,z,t)}{\partial t}+u(T)\frac{\partial T(x,y,z,t)}{\partial x}=\nu\left[\frac{\partial^2 T(x,y,z,t)}{\partial x^2}+\frac{\partial^2 T(x,y,z,t)}{\partial y^2}+\frac{\partial^2 T(x,y,z,t)}{\partial z^2}\right],$$
$$0<x<1,\quad 0<y<1,\quad 0<z<1,\quad t>0 \tag{11a}$$

with initial and homogeneous boundary conditions given by

$$T(x,y,z,0)=1,\quad 0\le x\le 1,\quad 0\le y\le 1,\quad 0\le z\le 1 \tag{11b}$$

$$T(0,y,z,t)=0;\quad T(1,y,z,t)=0,\quad t>0 \tag{11c,d}$$

$$\frac{\partial T(x,0,z,t)}{\partial y}=0;\quad T(x,1,z,t)=0,\quad t>0 \tag{11e,f}$$

$$\frac{\partial T(x,y,0,t)}{\partial z}=0;\quad T(x,y,1,t)=0,\quad t>0 \tag{11g,h}$$

and, for the present application, the nonlinear function $u(T)$ is taken as

$$u(T)=u_0+cT \tag{11i}$$

Results are also presented for the case with nonhomogeneous boundary conditions, in which the boundary condition given by Eq. (11c) is replaced by

$$T(0, y, z, t) = 1. \tag{12}$$

4.2 Conjugated Heat Transfer in Microchannels

There has been a marked research interest in improving the prediction and design tools for transport phenomena driven microsystems. In this context, Knupp et al. (2012, 2013a,b) have recently introduced an approach for the treatment of conjugated convection-conduction problems, based on the reformulation of the coupled problem into a single-domain model, which accounts for the heat transfer at both the fluid flow and the channel wall regions. By making use of coefficients represented as space variable functions, with abrupt transitions occurring at the fluid–wall interface, the mathematical model is fed with information concerning the two original domains of the problem. Then the GITT is employed with the integral transformation being constructed based upon an eigenvalue problem with space variable coefficients (Naveira-Cotta et al., 2009), thus incorporating all the information regarding the transition between the two original domains.

The problem here chosen for illustration, schematically represented in Fig. 1, involves a laminar incompressible internal flow of a Newtonian fluid between parallel plates, in steady state and undergoing convective heat transfer due to a prescribed temperature T_w, at the external face of the microchannel wall. The channel wall is considered to participate on the heat transfer problem through both transversal and axial heat conduction. The fluid flows with a known fully developed velocity profile $u_f(y)$, and with a uniform inlet temperature T_{in}. The flow is assumed to be hydrodynamically developed but thermally developing, with negligible viscous dissipation and temperature-independent physical properties.

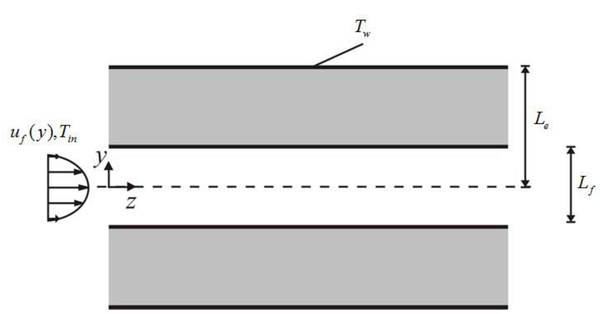

FIG. 1: Schematic representation of the conjugated heat transfer problem in a microchannel

We choose to formulate this problem using the single-domain formulation approach by making use of coefficients represented as space variable functions, with abrupt transitions occurring at the fluid–wall interface. Recalling the problem symmetry at $y = 0$, the formulation of the conjugated problem as a single region model is written as (Knupp et al., 2012)

$$u(y)w_f \frac{\partial T(y,z)}{\partial z} = k(y)\frac{\partial^2 T}{\partial z^2} + \frac{\partial}{\partial y}\left(k(y)\frac{\partial T}{\partial y}\right), \quad 0 < y < L_e, \quad 0 < z < z_\infty \tag{13a}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \quad T(L_e, z) = T_w, \quad 0 < z < z_\infty \tag{13b,c}$$

$$T(y,0) = T_{in}, \qquad \left.\frac{\partial T(y,z)}{\partial z}\right|_{z=z_\infty} = 0, \quad 0 < y < L_e \tag{13d,e}$$

with

$$u(y) = \begin{cases} u_f(y), & \text{if } 0 < y < L_f/2 \\ 0, & \text{if } L_f/2 < y < L_e \end{cases}, \quad k(y) = \begin{cases} k_f, & \text{if } 0 < y < L_f/2 \\ k_s, & \text{if } L_f/2 < y < L_e \end{cases} \tag{13f,g}$$

where w_f is the fluid heat capacity, k_s is the channel wall thermal conductivity, k_f is the fluid thermal conductivity, and $u_f(y)$ is the fully developed velocity profile. The following dimensionless groups are defined:

$$Z = \frac{z/D_h}{\mathrm{Re}\,\mathrm{Pr}} = \frac{z}{D_h \mathrm{Pe}}; \quad Y = \frac{y}{L_e}; \quad U = \frac{u}{u_{av}}; \quad \theta = \frac{T - T_{in}}{T_w - T_{in}}; \quad K = \frac{k}{k_f}; \quad \mathrm{Re} = \frac{u_{av} D_h}{\nu}; \quad \mathrm{Pr} = \frac{\nu}{\alpha}; \quad \mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr} = \frac{u_{av} D_h}{\alpha}; \quad \alpha = \frac{k_f}{w_f}; \quad \sigma = \frac{L_e}{L_f}; \quad Y_i = \frac{L_f}{2L_e} \tag{14a–k}$$

where the hydraulic diameter is given by $D_h = 2L_f$.

After making use of the groups given by Eqs. (14a)–(14k), one obtains the dimensionless formulation which is here rewritten already accounting for a pseudo-transient term in the formulation:

$$\frac{\partial \theta(Y,Z,t)}{\partial t} = -U(Y)\frac{\partial \theta}{\partial Z} + \frac{K(Y)}{\mathrm{Pe}^2}\frac{\partial^2 \theta}{\partial Z^2} + \frac{4}{\sigma^2}\frac{\partial}{\partial Y}\times\left(K(Y)\frac{\partial \theta}{\partial Y}\right), \quad 0 < Y < 1, \quad 0 < Z < Z_\infty, \quad t > 0 \tag{15a}$$

$$\theta(Y,Z,0)=\theta_{t=0},\ \ 0<Y<1,\ \ 0<Z<Z_\infty \quad \text{(15b)}$$

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0}=0,\quad \theta(1,Z,t)=1,$$
$$0<Z<Z_\infty,\qquad t>0 \quad \text{(15c,d)}$$

$$\theta(Y,0,t)=\theta_{Z=0}=0,\qquad \left.\frac{\partial\theta}{\partial Z}\right|_{Z=Z_\infty}=0,$$
$$0<Y<1,\qquad t>0 \quad \text{(15e,f)}$$

with

$$U(Y)=\begin{cases} U_f(Y), & \text{if}\quad 0<Y<Y_i \\ 0, & \text{if}\quad Y_i<Y<1 \end{cases},$$
$$K(Y)=\begin{cases} 1, & \text{if}\quad 0<Y<Y_i \\ k_s/k_f, & \text{if}\quad Y_i<Y<1 \end{cases} \quad \text{(15g,h)}$$

where the initial condition $\theta_{t=0}$ is preferably an estimate of the steady-state solution to accelerate convergence, or just any reasonable function, since we are only interested in the steady-state results. Figures 2(a) and 2(b) show the space variable coefficients with abrupt transition at the fluid–solid interface, $U(Y)$ and $K(Y)$, respectively.

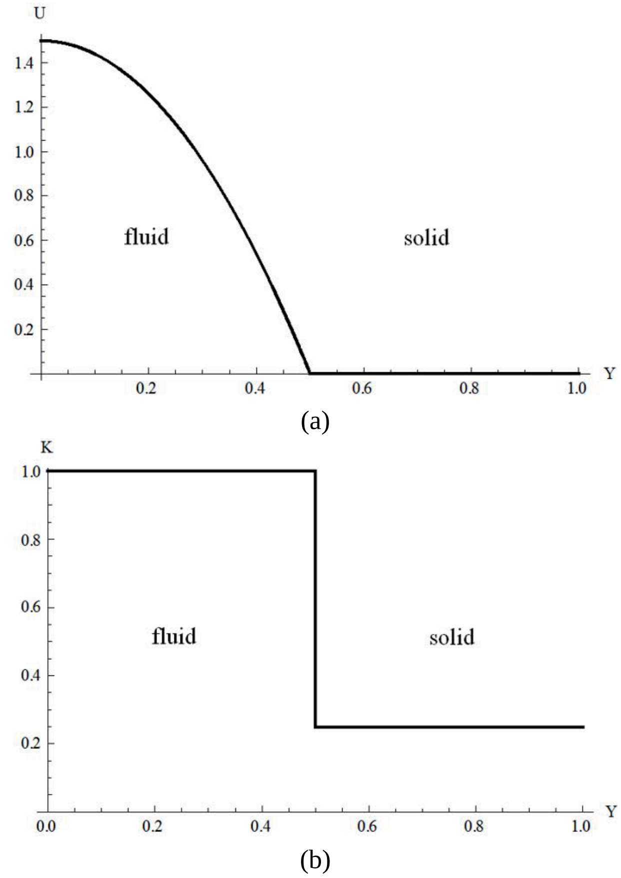

FIG. 2: Representation of the space variable coefficients as functions with abrupt transitions occurring at the fluid–solid wall interface: **(a)** $U(Y)$ and **(b)** $K(Y)$

5. RESULTS AND DISCUSSION

5.1 Nonlinear Burgers' Equation

In the analysis of the three-dimensional Burgers' equation, Eqs. (11) and (12), the following governing parameters values have been adopted (Cotta et al., 2013): u_0 = 1, c = 5, and ν = 1. The Burgers' equation formulation with homogenous boundary conditions is investigated first. In order to offer a benchmark solution, the UNIT code with total transformation was first employed with user-provided analytical integration of the transformed initial condition and nonlinear source term, and double checked against a dedicated GITT solution of the same problem (Cotta et al., 2013). Then, high truncation orders could be achieved with reduced computational cost so as to offer a reliable hybrid solution with four fully converged significant digits, as presented in (Cotta et al., 2013), where convergence is reached at truncation orders around N = 300 with the traditional squared eigenvalues reordering criterion. Table 1 illustrates the convergence behavior of the UNIT code with the partial transformation scheme, hereinafter called UNIT PT, in which the direction x has been chosen not to be integral transformed. The results are presented for a different number of terms in the eigenfunction expansion, N = 30, 35, 40, and 45, and fixed number of points in the Gaussian quadrature integration, M = {19,19}. In Table 1 the standard total transformation UNIT solution (as obtained from the last column in Table 1a of Cotta et al., 2013) is also presented, here called UNIT TT, obtained with user-provided analytical integration. The fully discrete solution is also presented, obtained by the method of lines from the NDSolve routine of the Mathematica system, under the same precision control with mesh refinement, as in the options employed in the UNIT PT solution. One should notice that, at the selected positions, the UNIT PT results are converged to the third significant digit for t = 0.02 and to all four digits shown for t = 0.1, with the expected improved convergence for larger values of t. It is also observed that the UNIT PT results agree with the NDSolve numerical results in three significant digits for t = 0.02 and t = 0.1.

TABLE 1: Convergence of UNIT code solution with the partial transformation scheme for a three-dimensional Burgers' equation with homogeneous boundary conditions (GITT with N = 30, 35, 40, or 45 terms and $M = \{19,19\}$ in the Gaussian quadrature integration)

$T = 0.02$, $y = 0.5$, $z = 0.5$						
x	N = 30	N = 35	N = 40	N = 45	UNIT TT[a]	NDSolve[b]
0.1	0.2780	0.2803	0.2807	0.2808	0.2798	0.2807
0.3	0.7354	0.7401	0.7407	0.7409	0.7396	0.7406
0.5	0.9291	0.9333	0.9336	0.9337	0.9331	0.9336
0.7	0.9127	0.9155	0.9153	0.9152	0.9156	0.9155
0.9	0.4898	0.4898	0.4892	0.4891	0.4905	0.4896
$T = 0.1$, $y = 0.5$, $z = 0.5$						
x	N = 30	N = 35	N = 40	N = 45	UNIT TT[a]	NDSolve[b]
0.1	0.04943	0.04945	0.04946	0.04946	0.04922	0.04947
0.3	0.1494	0.1494	0.1494	0.1494	0.1492	0.1494
0.5	0.2232	0.2232	0.2232	0.2232	0.2231	0.2233
0.7	0.2236	0.2236	0.2236	0.2236	0.2235	0.2236
0.9	0.1031	0.1031	0.1031	0.1031	0.1031	0.1031

[a] N = 300 terms and analytical integration and total transformation, from Table 1(a) of (Cotta et al., 2013);

[b] Mathematica 7.

These results, in comparison with those obtained by the UNIT TT, indicate a faster convergence rate of the UNIT PT solution, as expected, since three eigenfunction expansions are represented in the UNIT TT solution, while only two are represented in the UNIT PT solution. Nevertheless, it should be remembered that the UNIT PT solution demands additional numerical efforts per equation, since a system of one-dimensional PDEs is being solved, while for the UNIT TT scheme the total integral transformation process leads to a transformed ODE system. This may also help explaining the fact that the UNIT PT results are closer to the fully discrete ones than to the benchmark UNIT TT solution. In the UNIT PT, even considerably reducing the numerical effort when transforming from a three-dimensional partial differential system to a one-dimensional one, the numerical solution of the coupled PDE's transformed system still yields a considerable effect on the final computed results. Apparently, only under *a priori* imposed variable meshing or adaptive meshing refinement, the UNIT PT and the fully discrete approach would be able to provide final results of closer accuracy to the hybrid numerical–analytical solution with total transformation, here adopted as a benchmark. Figure 3 depicts

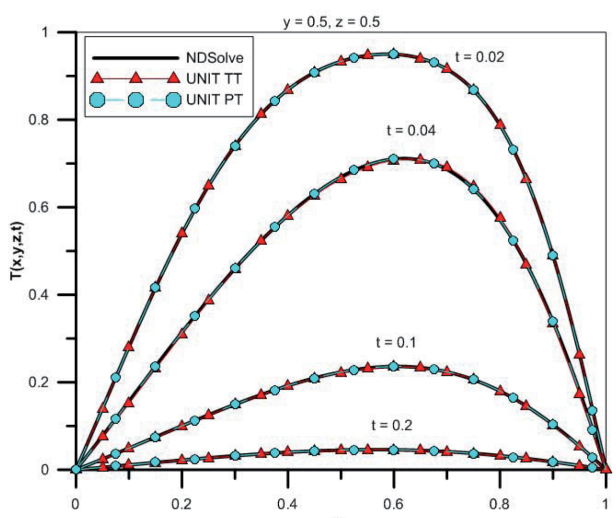

FIG. 3: Comparison between the UNIT PT and TT solutions and the NDSolve routine solution for the three-dimensional Burgers' equation problem with homogeneous boundary conditions

the graphical comparison of the UNIT PT and UNIT TT solutions, at different times, against those obtained with the NDSolve routine, where it is observed to be in excellent agreement between the three solutions throughout, to the graph scale.

Proceeding to the Burgers' equation problem with nonhomogeneous boundary conditions, Table 2 illustrates the convergence behavior of the UNIT solution with partial transformation, by varying the truncation order, N = 55, 60, 65, and 70, with a fixed number of points in the Gaussian quadrature integration, $M = \{19{,}19\}$. We point out that no filtering procedure was performed in this solution, since the direction with nonhomogeneous boundary conditions x is the one chosen not to be integral transformed. In these results, an excellent convergence behavior is observed at the selected positions, the results being converged to the fourth significant digit throughout. It should be stressed that, based on the truncation order selected by the user, the UNIT code is able to automatically provide a safe number of points in the Gaussian numerical integration procedure, in case the user does not wish to manually set this option.

Figure 4 brings the comparison between the solutions obtained with the UNIT PT and the UNIT TT with the user-provided filtering option, against the fully discrete NDSolve routine solution, along the x coordinate. In these results one may again confirm the excellent adherence between the UNIT solutions as well as with the NDSolve solution curves throughout the x variable domain.

5.2 Conjugated Heat Transfer in Microchannels

The dimensionless thermal conductivity has been calculated, motivated by an application with a microchannel etched on a polyester resin substrate (k_s = 0.16 W/m°C) and water as the working fluid (k_f = 0.64 W/m°C), so that k_s/k_f = 0.25 (Ayres et al., 2011; Knupp et al., 2013b). The single-domain approach combined with inte-

TABLE 2: Convergence of UNIT code solution with the partial transformation scheme for three-dimensional Burgers' equation with nonhomogeneous boundary conditions (GITT with N = 55, 60, 65, or 70 terms and $M = \{19{,}19\}$ in the Gaussian quadrature integration)

t = 0.04						
(x, y, z)	N = 55	N = 60	N = 65	N = 70	UNIT TT[a]	NDSolve[b]
(0.2,0.5,0.5)	0.9407	0.9409	0.9408	0.9408	0.9433	0.9408
(0.5,0.5,0.5)	0.8523	0.8523	0.8523	0.8523	0.8493	0.8523
(0.8,0.5,0.5)	0.5945	0.5945	0.5945	0.5945	0.5904	0.5948
(0.5,0.2,0.5)	0.9119	0.9119	0.9119	0.9119	0.9089	0.9119
(0.5,0.8,0.5)	0.4897	0.4896	0.4896	0.4897	0.4912	0.4896
(0.5,0.5,0.2)	0.9119	0.9119	0.9119	0.9119	0.9089	0.9119
(0.5,0.5,0.8)	0.4897	0.4897	0.4897	0.4897	0.4912	0.4896
t = 0.1						
(x, y, z)	N = 55	N = 60	N = 65	N = 70	UNIT TT[a]	NDSolve[b]
(0.2,0.5,0.5)	0.8798	0.8801	0.8799	0.8799	0.8805	0.8798
(0.5,0.5,0.5)	0.6400	0.6400	0.6400	0.6400	0.6340	0.6398
(0.8,0.5,0.5)	0.3365	0.3365	0.3365	0.3365	0.3307	0.3364
(0.5,0.2,0.5)	0.7457	0.7457	0.7457	0.7457	0.7402	0.7456
(0.5,0.8,0.5)	0.3282	0.3282	0.3282	0.3282	0.3279	0.3281
(0.5,0.5,0.2)	0.7457	0.7457	0.7457	0.7457	0.7402	0.7456
(0.5,0.5,0.8)	0.3282	0.3282	0.3282	0.3282	0.3279	0.3281

[a]UNIT code solution with total transformation and user-provided polynomial filter (N = 60, $M = \{8{,}8{,}10\}$);

[b]Mathematica 7.

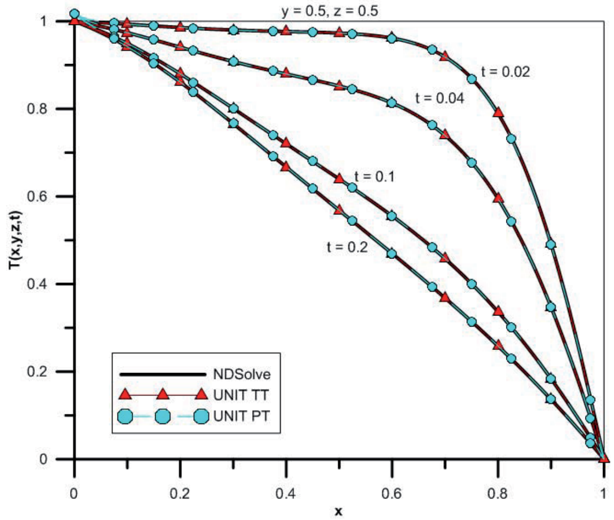

FIG. 4: Comparison between the UNIT PT and UNIT TT solutions vs the NDSolve routine solution for the Burgers' equation problem with nonhomogeneous boundary conditions, along x

gral transforms for the solution of conjugated heat transfer problems has been introduced, verified, and validated in Knupp et al. (2012), while the same problem considered here was previously tackled in Knupp et al. (2013a) with two alternative approaches, thus offering some benchmark results for the comparison of the UNIT solutions here presented. For the solution via the UNIT PT scheme, the simplest possible auxiliary eigenvalue problem was adopted, with constant coefficients, so as to lead to an auxiliary problem with analytical solution readily obtained from the DSolve routine of the Mathematica system, the automatic option of the UNIT code. In this least informative choice, the terms with abrupt spatial transition that are responsible for the information on the transition of the two regions (fluid stream and channel walls) are grouped into the source term. In the results below, we illustrate the solution obtained for the case with Pe = 0.5.

Table 3 presents the convergence behavior of the UNIT PT scheme with respect to the number of terms in the eigenfunction expansion, N = 10, 15, 20, 25, or 30, with fixed M = 96 points in the Gaussian quadrature integration. One may observe that the results are converged to the third or even up to the fourth significant digit with respect to the truncation order, at the selected positions. The comparison with the reference GITT values (Knupp et al., 2013a), which is fully converged to all digits shown, indicates a three-significant-digits agreement with the automatic UNIT PT solution at the selected positions.

Figure 5 depicts the comparison of the UNIT PT solution with the reference curves for the transversal temperature profiles for some different longitudinal positions along the flow, Z = 0.05, 0.1, 0.2, 0.3, 0.5, 0.75, 1.0, 1.5, and 4.5, for both the fluid and the channel wall regions. Despite the adoption of the simplest possible auxiliary eigenvalue problem, a very good graphical agreement between the automatic UNIT PT solution and the dedicated GITT solution is observed throughout. In fact, in order to investigate the behavior of the solution of the above single-domain formulation by a purely numerical method, we have attempted to solve the original partial differential problem through the NDSolve routine of the Mathematica package under automatic accuracy control. However, under default specifications, it was not able to accurately solve the problem with the intrinsic maximum refinement options. We have also tried to impose limits of refinement

TABLE 3: Convergence behavior of the conjugated problem solution with Pe = 0.5 obtained with the UNIT code via partial transformation with N = 10, 15, 20, 25, or 30 terms and M = 96 points in the Gaussian quadrature integration

N	Z = 0.1			Z = 0.2		
	Y = 0.3	Y = 0.6	Y = 0.9	Y = 0.3	Y = 0.6	Y = 0.9
10	0.06511	0.1247	0.4899	0.1328	0.2321	0.6920
15	0.06672	0.1236	0.4928	0.1330	0.2333	0.6921
20	0.06667	0.1233	0.4935	0.1328	0.2333	0.6920
25	0.06670	0.1231	0.4934	0.1330	0.2329	0.6920
30	0.06634	0.1230	0.4932	0.1329	0.2329	0.6920
GITT[a]	0.06626	0.1241	0.4935	0.1321	0.2348	0.6923

[a] GITT solution with analytical integration (Knupp et al., 2013a).

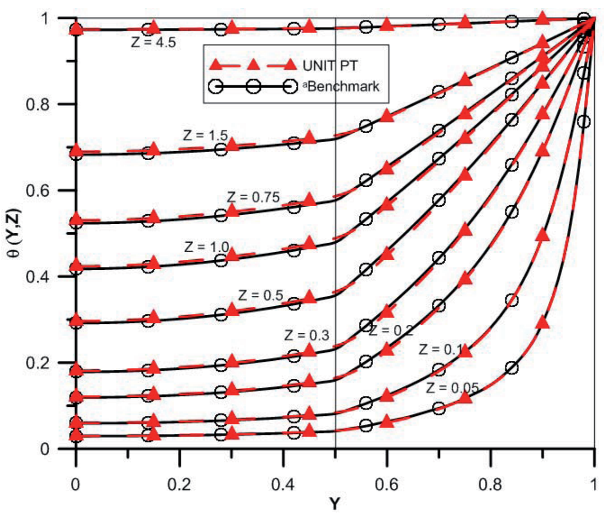

FIG. 5: Transversal temperature profiles in conjugated problem, calculated using the single-domain formulation via the UNIT PT scheme, in comparison with benchmark GITT results. [a](Knupp et al., 2013a) for Pe = 0.5

higher than the default options, up to the maximum available computation capability, and satisfactory results were still not obtained, reconfirming the relative merits of the hybrid solution approach.

6. CONCLUSIONS

This work compared the total and partial transformation schemes featured into the unified integral transform algorithm, implemented in the so-called UNIT code, as part of the ongoing project related to the progressive construction of an open-source symbolic-numerical computational code for finding solutions of partial differential systems based on the generalized integral transform technique (GITT). The implementations have been compared for nonlinear three-dimensional formulations of Burgers' equation, where it has been illustrated that the partial transformation scheme can be straightforward and accurate, leading to a hybrid solution path that may offer advantages inherent to the analytical development in combination with the powerful numerical solution procedure available. As an application illustration, the solution of a conjugated conduction–convection heat transfer problem of laminar flow inside parallel plate microchannels has been demonstrated. This problem has been reformulated by means of a single-domain approach, recently introduced with the main objective of simplifying the conjugated heat transfer analysis, without requiring any sort of iterative procedure. Again, the UNIT code under the partial transformation scheme was capable of recovering with accuracy and mild computational cost, the benchmark results obtained with a dedicated GITT solution with analytical integration.

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REFERENCES

Almeida, A. R. and Cotta, R. M., Integral transform methodology for convection-diffusion problems in petroleum reservoir engineering, *Int. J. Heat Mass Transfer*, vol. **38**, no. 18, pp. 3359–3367, 1995.

Alves, L. S. B., Cotta, R. M., and Pontes, J., Stability analysis of natural convection in porous cavities through integral transforms, *Int. J. Heat Mass Transfer*, vol. **45**, pp. 1185–1195, 2002.

Aparecido, J. B. and Cotta, R. M., Fully developed laminar flow in trapezoidal ducts, *The 9th Brazilian Congress of Mechanical Engineering*, IX COBEM, Florianópolis, SC, vol. **1**, pp. 25–28, Dec. 7–11, 1987.

Aparecido, J. B., Cotta, R. M., and Ozisik, M. N., Analytical solutions to two-dimensional diffusion type problems in irregular geometries, *J. Franklin Inst.*, vol. **326**, pp. 421–434, 1989.

Ayres, J. V. C., Knupp, D. C., Naveira-Cotta, C. P., Ferreira, L. O. S., and Cotta, R. M., Evaluation of infrared thermography experimental analysis of a single micro-channel heat spreader, *The 21st Intl. Congress of Mechanical Engineering*, COBEM-2011, ABCM, Natal, RN, Brazil, Oct. 24–28, 2011.

Baohua, C. and Cotta, R. M., Integral transform analysis of natural convection in porous enclosures, *Int. J. Num. Methods Fluids*, vol. **17**, pp. 787–801, 1993.

Bayazitoglu, Y. and Ozisik, M. N., On the solution of Graetz-type problems with axial conduction, *Int. J. Heat Mass Transfer*, vol. **23**, pp. 1399–1402, 1980.

Bogado Leite, S. Q., Ozisik, M. N., and Verghese, K., On the solution of linear diffusion problems in media with moving boundaries, *Nucl. Sci. Eng.*, vol. **76**, pp. 345–350, 1980.

Bogado Leite, S. Q., Ozisik, M. N., and Verghese, K., Diffusion in a two-region slab with an eroding boundary, *Int. J. Heat Mass Transfer*, vol. **25**, pp. 1191–1197, 1982.

Carslaw, H. S. and Jaeger, D. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1947.

Carvalho, T. M. B., Cotta, R. M., and Mikhailov, M. D., Flow development in the entrance regions of ducts, *Commun. Numer. Methods Eng.*, vol. **9**, pp. 503–509, 1993.

Cotta, R.M., Diffusion in media with prescribed moving boundaries: Application to metals oxidation at high temperatures, *Proc. of the 2nd Latin American Congress of Heat and Mass Transfer*, São Paulo, Brazil, vol. **1**, pp. 502–513, May 1986a.

Cotta, R. M., Steady-state diffusion with space-dependent boundary condition parameters, *Proc. of the 1st National Meeting of Thermal Sciences*, ENCIT 86, pp. 163–166, Rio de Janeiro, Brazil, Dec. 1986b.

Cotta, R. M., Hybrid numerical-analytical approach to nonlinear diffusion problems, *Numer. Heat Transfer, Part B*, vol. **127**, pp. 217–226, 1990.

Cotta, R. M., *Integral Transforms in Computational Heat and Fluid Flow*, CRC Press, Boca Raton, FL, 1993.

Cotta, R. M., The integral transform method in computational heat and fluid flow, Special Keynote Lecture, *Proc. of the 10th Intl. Heat Transfer Conf.*, Brighton, United Kingdom, SK-3, vol. **1**, pp. 43–60, Aug. 1994a.

Cotta, R. M., Benchmark results in computational heat and fluid flow: The integral transform method, *Int. J. Heat Mass Transfer* (Invited Paper), vol. **37**, pp. 381–394, 1994b.

Cotta, R. M. (Ed.), *The Integral Transform Method in Thermal and Fluids Sciences and Engineering*, Begell House, New York, 1998.

Cotta, R. M. and Carvalho, T. M. B., Hybrid analysis of boundary layer equations for internal flow problems, *The 7th Intl. Conf. on Numerical Methods in Laminar and Turbulent Flow*, Stanford, CA, Part 1, pp. 106–115, July 1991.

Cotta, R. M. and Mikhailov, M. D., *Heat Conduction: Lumped Analysis, Integral Transforms, Symbolic Computation*, Wiley-Interscience, Chichester, United Kingdom, 1997.

Cotta, R. M. and Mikhailov, M. D., Hybrid methods and symbolic computations, in *Handbook of Numerical Heat Transfer*, 2nd ed., Chap. 16, W.J. Minkowycz, E.M. Sparrow, and J.Y. Murthy, Eds., John Wiley, New York, 2006.

Cotta, R. M. and Orlande, H. R. B., Hybrid approaches in heat and mass transfer: A Brazilian experience with applications in national strategic projects, *Heat Transfer Eng.*, Invited Editorial, vol. **24**, no. 4, pp. 1–5, 2003.

Cotta, R. M. and Ozisik, M. N., Transient forced convection in laminar channel flow with stepwise variations of wall temperature, *ASME Winter Annual Meeting*, Miami, FL, Nov. 1985; also in *Can. J. Chem. Eng*., vol. **64**, pp. 734–742, 1986.

Cotta, R. M. and Ozisik, M. N., Laminar forced convection in ducts with periodic variation of inlet temperature, *Int. J. Heat Mass Transfer*, vol. **29**, no. 10, pp. 1495–1501, 1986.

Cotta, R. M. and Ozisik, M. N., Diffusion problems with general time-dependent coefficients, *Rev. Bras. Ciências Mecânicas*, vol. **9**, no. 4, pp. 269–292, 1987.

Cotta, R. M., Ungs, M. J., and Mikhailov, M. D., Contaminant transport in finite fractured porous medium: Integral transforms and lumped-differential formulations, *Ann. Nucl. Energy*, vol. **30**, no. 3, pp. 261–285, 2003.

Cotta, R. M., Quaresma, J. N. N., Sphaier, L. A., and Naveira-Cotta, C. P., Unified integral transform approach in the hybrid solution of multidimensional nonlinear convection-diffusion problems, *The 14th Intl. Heat Transfer Conf.*, Washington, DC, Aug. 2010.

Cotta, R. M., Knupp, D. C., Naveira-Cotta, C. P., Sphaier, L. A., and Quaresma, J. N. N., Unified integral transform algorithm for solving multidimensional nonlinear convection-diffusion problems, *Numer. Heat Transfer, Part A*, vol. **63**, no. 11, pp. 840–866, 2013.

Debnath, L. and Bhatta, D., *Integral Transforms and Their Applications*, 2nd ed., Taylor and Francis Group, LLC, London, 2007.

Diniz, A. J., Aparecido, J. B., and Cotta, R. M., Heat conduction with ablation in a finite slab, *Int. J. Heat Technol.*, vol. **8**, pp. 30–43, 1990.

Fourier, J., *Thorie Analytique de la Chaleur*, Firmin Didot Père et Fils, Paris, 1822.

Grinberg, G. A., Selected problems of mathematical theory of electrical and magnetic effects, *Akad. Nauk. SSSR*, 1948.

Guedes, R. O. C. and Cotta, R. M., Periodic Laminar forced convection within ducts including wall heat conduction effects, *Int. J. Eng. Sci*., vol. **29**, no. 5, pp. 535–547, 1991.

Guedes, R. O. C., Cotta, R. M., and Brum, N. C. L., Conjugated heat transfer in laminar flow between parallel-plates channel, *The 10th Brazilian Congress of Mechanical Engineering*, X COBEM, Rio de Janeiro, pp. 285–288, Dec. 1989.

Knupp, D. C., Naveira-Cotta, C. P., Ayres, J. V. C., Orlande, H. R. B., and Cotta, R. M., Experimental-theoretical analysis in a transient heat conduction setup via infrared thermography and unified integral transforms, *Int. Rev. Chem. Eng., IRECHE*, vol. **2**, no. 6, pp. 736–747, 2010.

Knupp, D. C., Naveira-Cotta, C. P., and Cotta, R. M., Theoretical analysis of conjugated heat transfer with a single domain formulation and integral transforms, *Int. Commun. Heat Mass Transfer*, vol. **39**, no. 3, pp. 355–362, 2012.

Knupp, D. C., Cotta, R. M., and Naveira-Cotta, C. P., Theoretical-experimental analysis of conjugated heat transfer in micro-channel heat spreaders, Paper no. HT2013-17818, *Proc. of the ASME 2013 Summer Heat Transfer Conf.*, HT2013, Minneapolis, MN, July 14–19, 2013a.

Knupp, D. C., Naveira-Cotta, C. P., and Cotta, R. M., Conjugated convection-conduction analysis in microchannels with axial diffusion effects and a single domain formulation, *ASME J. Heat Transfer*, Paper no. 091401-1, vol. **135**, 2013b.

Knupp, D. C., Naveira-Cotta, C. P., and Cotta, R. M., Theoretical-experimental analysis of conjugated heat transfer in nanocomposite heat spreaders with multiple microchannels, *Int. J. Heat Mass Transfer*, vol. **74**, pp. 306–318, 2014a.

Knupp, D. C., Cotta, R. M., Naveira-Cotta, C. P., and Kakaç, S., Transient conjugated heat transfer in microchannels: Integral transforms with single domain formulation, *Int. J. Therm. Sci*., In press, DOI: 10.1016/j.ijthermalsci.2014.04.017, 2014b.

Koshlyakov, N. S., *Fundamental Differential Equations of Mathematical Physics*, Moscow, 1936.

Leal, M. A., Machado, H. A., and Cotta, R. M., Integral transform solutions of transient natural convection in enclosures with variable fluid properties, *Int. J. Heat Mass Transfer*, vol. **43**, no. 21, pp. 3977–3990, 2000.

Luikov, A. V., *Analytical Heat Diffusion Theory*, Academic Press, New York, 1968.

Luikov, A. V., *Heat and Mass Transfer*, Mir Publishers, Moscow, 1980.

Luz Neto, H., Quaresma, J. N. N., and Cotta, R. M., Natural convection in three-dimensional porous cavities: Integral transform method, *Int. J. Heat Mass Transfer*, vol. **45**, no. 14, pp. 3013–3032, 2002.

Machado, H. A. and Cotta, R. M., Analysis of internal convection with variable physical properties via integral transformation, *Numer. Heat Transfer, Part A*, vol. **36**, no. 7, pp. 699–724, 1999.

Mikhailov, M. D., *Nonstationary Temperature Fields in Skin*, Energiya, Moscow, 1967.

Mikhailov, M.D., General solution of the heat equation of finite regions, *Int. J. Eng. Sci*., vol. **10**, pp. 577–591, 1972.

Mikhailov, M. D., General solution of the diffusion equations coupled at boundary conditions, *Int. J. Heat Mass Transfer*, vol. **16**, pp. 2155–2164, 1973a.

Mikhailov, M. D., General solution of the coupled diffusion equations, *Int. J. Eng. Sci*., vol. **11**, pp. 235–241, 1973b.

Mikhailov, M. D., On the solution of the heat equation with time dependent coefficient, *Int. J. Heat Mass Transfer*, vol. **18**, pp. 344–345, 1975.

Mikhailov, M. D., Splitting up of heat-conduction problems, *Int. Commun. Heat Mass Transfer*, vol. **4**, pp. 163–166, 1977a.

Mikhailov, M. D., General solution of diffusion processes in solid-liquid extraction, *Int. J. Heat Mass Transfer*, vol. **20**, pp. 1409–1415, 1977b.

Mikhailov, M. D. and Ozisik, M. N., An alternative general solution of the steady-state heat diffusion equation, *Int. J. Heat Mass Transfer*, vol. **23**, pp. 609–612, 1980.

Mikhailov, M. D. and Ozisik, M. N., A general solution of solute diffusion with reversible reaction, *Int. J. Heat Mass Transfer*, vol. **24**, pp. 81–87, 1981.

Mikhailov, M. D. and Ozisik, M. N., *Unified Analysis and Solutions of Heat and Mass Diffusion*, John Wiley, New York and Dover Publications, New York, 1984.

Mikhailov, M. D. and Shishedjiev, B. K., Coupled at boundary mass or heat transfer in entrance concurrent flow, *Int. J. Heat Mass Transfer*, vol. **19**, pp. 553–557, 1976.

Mikhailov, M. D. and Vulchanov, N. L., A computational procedure for Sturm-Liouville problems, *J. Comput. Phys*., vol. **50**, pp. 323–336, 1983.

Mikhailov, M. D., Ozisik, M. N., and Shishedjiev, B. K., Diffusion in heterogeneous media, *J. Heat Transfer*, vol. **104**, pp. 781–787, 1982.

Naveira-Cotta, C. P., Cotta, R. M., Orlande, H. R. B., and Fudym, O., Eigenfunction expansions for transient diffusion in heterogeneous media, *Int. J. Heat Mass Transfer*, vol. **52**, pp. 5029–5039, 2009.

Olçer, N. Y., On the theory of conductive heat transfer in finite regions, *Int. J. Heat Mass Transfer*, vol. **7**, pp. 307–314, 1964.

Olçer, N. Y., Note on the general solution of the heat equation, *Q. J. Appl. Math*., vol. **24**, p. 380, 1967.

Ozisik, M. N., *Boundary Value Problems of Heat Conduction*, International Textbooks Co., Scranton, PA, 1968.

Ozisik, M. N., *Heat Conduction*, John Wiley, New York, 1980.

Ozisik, M. N. and Guçeri, S. I., A variable eigenvalue approach to the solution of phase change problems, *Can. J. Chem. Eng*., vol. **55**, pp. 145–148, 1977.

Ozisik, M. N. and Murray, R. L., On the solution of linear diffusion problems with variable boundary condition parameters, *J. Heat Transfer*, vol. **96c**, pp. 48–51, 1974.

Pereira, L. M., Perez-Guerrero, J. S., Brazão, N., and Cotta, R. M., Compressible flow and heat transfer in ultracentrifuges: Hybrid analysis via integral transforms, *Int. J. Heat Mass Transfer*, vol. **45**, no. 1, pp. 99–112, 2002.

Perez Guerrero, J. S. and Cotta, R. M., Integral transform solution for the lid-driven cavity flow problem in streamfunction-only formulation, *Int. J. Numer. Methods Fluids*, vol. **15**, pp. 399–409, 1992.

Perez Guerrero, J. S., Quaresma, J. N. N., and Cotta, R. M., Simulation of laminar flow inside ducts of irregular geometry using integral transforms, *Comput. Mech.*, vol. **25**, no. 4, pp. 413–420, 2000.

Quaresma, J. N. N. and Cotta, R. M., Integral transform method for the Navier-Stokes equations in steady three-dimensional flow, *Proc. of the 10th ISTP, Intl. Symposium on Transport Phenomena*, Kyoto, Japan, pp. 281–287, Nov. 1997.

Ribeiro, J. W., Cotta, R. M., and Mikhailov, M. D., Integral transform solution of Luikov's equations for heat and mass transfer in capillary porous media, *Int. J. Heat Mass Transfer*, vol. **36**, no. 18, pp. 4467–4475, 1993.

Santos, C. A. C., Quaresma, J. N. N., and Lima, J. A., eds., Convective heat transfer in ducts: The integral transform approach, *ABCM Mech. Sci. Ser.*, Editora E-Papers, Rio de Janeiro, 2001.

Serfaty, R. and Cotta, R. M., Integral transform solutions of diffusion problems with nonlinear equation coefficients, *Int. Commun. Heat Mass Transfer*, vol. **17**, no. 6, pp. 851–864, 1990.

Serfaty, R. and Cotta, R. M., Hybrid analysis of transient nonlinear convection-diffusion problems, *Int. J. Numer. Methods Heat Fluid Flow*, vol. **2**, pp. 55–62, 1992.

Sphaier, L. A. and Cotta, R. M., Integral transform analysis of multidimensional eigenvalue problems within irregular domains, *Numer. Heat Transfer, Part B*, vol. **38**, pp. 157–175, 2000.

Sphaier, L. A. and Cotta, R. M., Integral transform solutions of diffusion and eigenvalue problems within arbitrarily shaped domains, *Proc. of the 12th Intl. Heat Transfer Conf.*, IHTC 12, Grenoble, France, Paper no. 0908, Aug. 2002.

Sphaier, L. A., Cotta, R. M., Naveira-Cotta, C. P., and Quaresma, J. N. N., The UNIT algorithm for solving one-dimensional convection-diffusion problems via integral transforms, *Int. Commun. Heat Mass Transfer*, vol. **38**, pp. 565–571, 2011.

Wolfram, S., *The Mathematica Book*, Wolfram Media, Cambridge, MA, 2005.

Yener, Y. and Ozisik, M. N., On the solution of unsteady heat conduction in multi-region media with time dependent heat transfer coefficient, *Proc. of the 5th Intl. Heat Transfer Conf.*, Tokyo, Japan, pp. 188–192, Sept. 3–7, 1974.