Rapid Communications

Vortex gas modeling of turbulent circulation statistics

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Statistical properties of circulation encode relevant information about the multiscale structure of turbulent cascades. Recent massive computational efforts have posed challenging theoretical issues, such as the dependence of circulation moments upon Reynolds numbers and length scales, and the specific shape of the heavy-tailed circulation probability distribution functions. We address these focal points in an investigation of circulation statistics for planar cuts of three-dimensional flows. The model introduced here borrows ideas from the structural approach to turbulence, whereby turbulent flows are depicted as dilute vortex gases, combined with the standard Obukhov-Kolmogorov phenomenological framework of small-scale intermittency. We are able to reproduce, in this way, key statistical features of circulation, in close agreement with empirical observations compiled from direct numerical simulations.

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I. INTRODUCTION

Drawing analogies with the Wilson loop strategy to tackle the quark confinement problem [1,2], Migdal introduced, some 25 years ago, alternative circulation functional methods to the context of fully developed turbulence [3]. The subject of turbulent circulation has now been vigorously revived both on the theoretical and the numerical fronts. Interesting ideas have reached firmer ground, as the area law for the probability distribution function of circulation [4–6], whereas unexpected phenomena have been additionally discovered, as the bifractal scaling behavior of circulation moments [5].

In this work, focused on the problem of isotropic and homogeneous turbulence, we define the circulation variable simply as

$$\Gamma_R \equiv \int_{\mathcal{D}} d^2\mathbf{r}\, \omega(\mathbf{r}), \tag{1.1}$$

where $\mathcal{D}$ is a circular domain of radius R which lies in a plane γ and $\omega(\mathbf{r})$ is the component of the vorticity field which is normal to γ (an arbitrary orientation is chosen). Our aim is to explore the sensitivity of (1.1) to the presence of vortex tubes—so clearly identified in turbulent flows since the early 1990s [7–9]—as a natural modeling perspective to account for relevant empirical findings.

Kolmogorov's phenomenological description of turbulence (K41) [10] suggests that if R is far from the energy injection and the Kolmogorov dissipative scales, L and η_K, respectively, the only other relevant physical parameter related to the statistical behavior of velocity fluctuations at scale R is the energy dissipation rate per unit mass, ϵ. We define, as usual, $\eta_K = (\nu^3/\epsilon)^{1/4}$, where ν is the kinematic viscosity of the fluid. Straightforward dimensional analysis leads, then, to circulation moments

$$\langle \Gamma_R^p \rangle \sim \epsilon^{p/3} R^{4p/3}. \tag{1.2}$$

Numerical simulations show that (1.2) is a reasonable approximation for low order moments, but it noticeably fails for $p > 4$, as expected from the existence of intermittent velocity fluctuations [5]. Our discussion of this issue brings together phenomenological views of turbulence that have so far been developed as two almost disconnected approaches in the literature: from one side, turbulence is viewed as a multiplicative cascade process across length scales; from another side, as the flow regime produced by a vorticity field dominantly organized in the form of vortex tubes [11]. It turns out, as we will see, that the integration of these two complementary pictures of turbulence is particularly important for a deeper understanding of circulation statistics.

This Rapid Communication is organized as follows. In Sec. II, we introduce, resorting on phenomenological inputs and heuristic arguments, a statistical model for the fluctuations of the circulation variable taken on planar contours. In Sec. III, we explore the proposed model to evaluate circulation moments, its inertial range scaling exponents, and to derive an analytical expression for the circulation probability distribution functions (cPDFs). We also perform supporting comparisons with numerical results. In Sec. IV, finally, we summarize our findings and point out directions of further research.

II. MODEL DEFINITIONS

Having in mind a reinterpretation of the scaling law (1.2) within the structural context, where it is assumed that most of the turbulent kinetic energy is generated by vortex tubes [8], consider the second-order moment of Γ_R, $\langle \Gamma_R^2 \rangle = \int_{\mathcal{D}} d^2\mathbf{r} \int_{\mathcal{D}} d^2\mathbf{r}' \langle \omega(\mathbf{r})\omega(\mathbf{r}')\rangle$. Taking this expression into account and noticing that the K41 prediction for order $p = 2$ can be reshuffled as

$$\langle \Gamma_R^2 \rangle \sim \left(\frac{R}{\eta_K}\right)^4 \left[\sqrt{\frac{\epsilon}{\nu}}\eta_K^2\right]^2 \left(\frac{\eta_K}{R}\right)^{4/3}, \tag{2.1}$$

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one may suggest that circulation is effectively produced by (i) a number $N \propto (R/\eta_K)^2$ of planar vortices which have (ii) rms vorticities of the order of $\sqrt{\epsilon/\nu}$, (iii) core sizes of linear dimensions of the order of η_K, and (iv) carry elementary circulations which are correlated at separation distance r as $\sim 1/r^{4/3}$, for $r \gg \eta_K$.

The deconstruction of (2.1) into the above points (i)–(iv) provides the main phenomenological motivation for setting up a two-dimensional vortex model of circulation fluctuations which could comprise, after refinements, anomalous scaling exponents. There are, however, important conceptual and technical issues that have to be dealt with, in order to proceed with this structural line of reasoning, which we discuss in the following sections.

A. Two-dimensionally projected vorticity field

A clear issue is the apparent dichotomy between the acknowledged existence of three-dimensional vortex tubes and the introduction of planar vortices in the modeling definitions. A possible solution to this problem is to take the positions of the effective planar vortices as the intersections of vortex tubes with the plane γ that contains the circular domain $\mathcal{D}$. Now, figuring out the large ensemble of three-dimensional flow configurations conditioned by a fixed spatial distribution of N intersecting vortex tubes in γ at positions $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$, it is clear that the vorticity vector at position $\mathbf{r}$ in γ can be represented as the conditioned random field

$$\boldsymbol{\omega}(\mathbf{r}) = \boldsymbol{\omega}(\mathbf{r}|\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N), \tag{2.2}$$

which is negligible if probed far enough from the planar vortex spots, that is, at positions $\mathbf{r}$ such that $|\mathbf{r} - \mathbf{r}_i| \gg \eta_K$ for $i = 1, 2, \dots, N$. The physical picture addressed here is illustrated in Fig. 1.

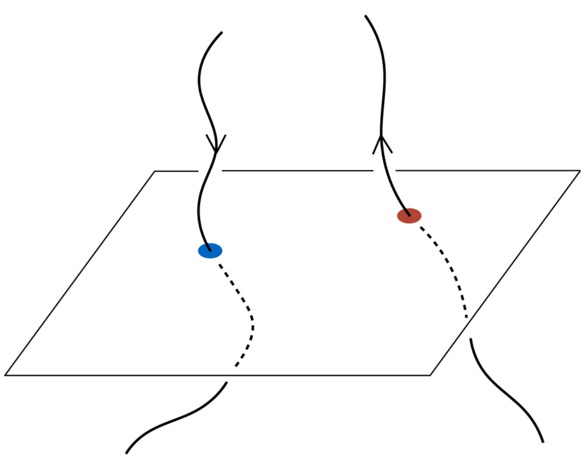

FIG. 1. Two thin vortex tubes with opposite orientations cross a plane and are associated, in this particular example, to a planar vortex-antivortex pair (red and blue spots, respectively) where vorticity is concentrated. In more general terms, for any given fixed configuration of planar vortices defined by means of a similar construction, there is a large statistical ensemble of three-dimensional vortex tubes that produce a (conditioned) random vorticity field on the plane.

B. Cascaded vorticity intensities

Relying upon the fact that $\boldsymbol{\omega}(\mathbf{r})$ is the superposition of the vorticity fields produced by a random system of three-dimensional vortex tubes, it is tempting to evoke the central limit theorem in some of its functional generalizations [12], to take $\boldsymbol{\omega}(\mathbf{r})$ as a Gaussian random field. This is actually a meaningful hint, but some care is necessary on this point, and here comes a second modeling difficulty: We have to include intermittent fluctuations of the energy dissipation rate in our arguments. In the Obukhov-Kolmogorov phenomenology of intermittency (OK62) [13,14] and its subsequent developments [11,15], local dissipation is described, to very good approximation, as a lognormal, long-ranged correlated field. Considering the modeling points (ii), (iii), and (iv), introduced above, we collect all the pieces of information brought to the discussion so far, to write

$$\omega(\mathbf{r}) \propto \sum_{i=1}^{N} g_\eta(\mathbf{r} - \mathbf{r}_i)\xi(\mathbf{r}_i)\tilde{\omega}(\mathbf{r}_i). \tag{2.3}$$

Here $g_\eta(\mathbf{r}) \equiv \exp[-\mathbf{r}^2/(2\eta^2)]$ introduces Gaussian envelopes for the planar vortices of typical width η which, in consonance with [16], are taken proportional to η_K, i.e., $\eta = a\eta_K$ with a being a positive modeling parameter. The field $\tilde{\omega}(\mathbf{r})$ is a scalar Gaussian random field with vanishing mean and correlator $\langle\tilde{\omega}(\mathbf{r})\tilde{\omega}(\mathbf{r}')\rangle \sim 1/|\mathbf{r} - \mathbf{r}'|^{4/3}$ for $|\mathbf{r} - \mathbf{r}'| \gg \eta_K$, while $\xi(\mathbf{r}) \equiv \xi_0\sqrt{\epsilon(\mathbf{r})/\epsilon_0}$, where ξ_0 is an additional positive parameter and $\epsilon(\mathbf{r})$, to be modeled within the OK62 framework, is the energy dissipation rate at position $\mathbf{r}$, which has mean value $\epsilon_0 = \langle\epsilon(\mathbf{r})\rangle$. We call attention, in connection with Eq. (2.3), to the general fact that the statistical dependence of velocity gradients with the square root of the dissipation field is the usual way to extend the OK62 description of intermittency to the dissipative scale region [17–19]. We take $\xi(\mathbf{r})$ and $\tilde{\omega}(\mathbf{r})$ to be dimensionless quantities. Without loss of generality, the variance of $\tilde{\omega}(\mathbf{r})$ is prescribed to unity.

C. Homogeneous vortex distributions

The positions and the number N of vortices in (2.3) are of course random quantities that depend on the shape and the size of the planar domain crossed by the vortex tubes. A natural modeling choice, as a first approximation, and inspired from visualizations of vortex tube structures [7–9], is to conjecture that vortices are randomly distributed over γ with a Poissonian surface density whose mean we denote as $\bar{\sigma} \equiv \langle\sigma(\mathbf{r})\rangle$. We do not claim that sub- or super-Poissonian density fluctuations, due to short-distance volume exclusion effects [20] or energy-reducing vortex-antivortex pairings [21], respectively, should be completely neglected. Instead, we assume that non-Poissonian density fluctuations are likely to have a subdominant role in the spatial organization of vortices.

Introducing a prefactor with dimensions of vorticity, we rewrite (2.3) in the more analytically convenient continuous version,

$$\omega(\mathbf{r}) = \sqrt{\frac{\epsilon_0}{3\nu}} \int d^2\mathbf{r}' g_\eta(\mathbf{r} - \mathbf{r}')\xi(\mathbf{r}')\tilde{\omega}(\mathbf{r}')\sigma(\mathbf{r}'). \tag{2.4}$$

Equation (2.3) can be in fact derived from (2.4) by taking

$$\sigma(\mathbf{r}) = \sum_i \delta^2(\mathbf{r} - \mathbf{r}_i). \quad (2.5)$$

To clarify the choice of the vorticity prefactor in (2.4), just note that $\langle \omega(\mathbf{r})^2 \rangle = \langle \boldsymbol{\omega}(\mathbf{r})^2 \rangle / 3 = \epsilon_0/(3\nu)$ due to isotropy [22].

Since direct numerical simulations indicate that the volume occupied by vortex tubes is a very small fraction of the total fluid volume [8], we may employ a dilute vortex gas approximation to represent density fluctuations as $\sigma(\mathbf{r}) = \bar{\sigma} + \phi(\mathbf{r})$, with $\langle \phi(\mathbf{r}) \rangle = 0$ and, up to fourth order,

$$\langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) \rangle = \bar{\sigma}\delta_{12}, \quad (2.6)$$

$$\langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3) \rangle = \bar{\sigma}\delta_{12}\delta_{13}, \quad (2.7)$$

$$\langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3)\phi(\mathbf{r}_4) \rangle = \bar{\sigma}\delta_{12}\delta_{13}\delta_{14} + \bar{\sigma}^2(\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23}), \quad (2.8)$$

where we have used the notation $\delta_{ij} = \delta^2(\mathbf{r}_i - \mathbf{r}_j)$ [23].

A smooth cutoff-regularized expression for the Gaussian random field $\tilde{\omega}(\mathbf{r})$ proves to be of great analytical help in the evaluation of vorticity correlation functions. We just mean that $\tilde{\omega}(\mathbf{r})$ is given as

$$\tilde{\omega}(\mathbf{r}) = \frac{\eta_K^{2/3}}{\sqrt{2\pi\Gamma\left(\frac{4}{3}\right)}} \int d^2\mathbf{k}\, \psi(\mathbf{k}) k^{-1/3} \exp\left(i\mathbf{k}\cdot\mathbf{r} - k\frac{\eta_K}{2}\right), \quad (2.9)$$

where $\psi(\mathbf{k})$ is a complex Gaussian random field of zero mean and correlator $\langle \psi(\mathbf{k}_1)\psi(\mathbf{k}_2) \rangle = \delta^2(\mathbf{k}_1 - \mathbf{k}_2)$.

D. Effective coarse-graining modeling

The formulation addressed by Eq. (2.4) becomes in fact cumbersome if one is interested to compute moments of order $p > 2$ for the circulation variable. The reason is that $\xi(\mathbf{r})$ is not a Gaussian random field. A pragmatic phenomenological solution to this problem comes from an alternative way of performing the circulation integral. Considering, initially, that $R \gg \eta_K$, we obtain, from Eqs. (1.1) and (2.4), the asymptotic approximation

$$\Gamma_R = 2\pi\eta^2 \sqrt{\frac{\epsilon_0}{3\nu}} \int_{\mathcal{D}} d^2\mathbf{r}\, \xi(\mathbf{r})\tilde{\omega}(\mathbf{r})\sigma(\mathbf{r}). \quad (2.10)$$

Since $\xi(\mathbf{r})$ is a positive-definite quantity, it is not difficult to show that there is necessarily a point $\mathbf{r}_0 \in \mathcal{D}$ such that

$$\Gamma_R = 2\pi\eta^2 \sqrt{\frac{\epsilon_0}{3\nu}} \tilde{\omega}(\mathbf{r}_0)\sigma(\mathbf{r}_0) \int_{\mathcal{D}} d^2\mathbf{r}\, \xi(\mathbf{r}). \quad (2.11)$$

Owing, now, to the fact that $\xi(\mathbf{r})$ is long-range correlated and that the probability measures for the $\tilde{\omega}(\mathbf{r})$ and $\sigma(\mathbf{r})$ fields are translation invariant, we expect $\mathbf{r}_0$ to be a random variable uniformly distributed over the domain $\mathcal{D}$, if flow configurations are conditioned to fixed $\mathcal{I} \equiv \int_{\mathcal{D}} d^2\mathbf{r}\xi(\mathbf{r})$. Therefore, Γ_R, as defined in (2.11), is the same as its average over fluctuations of $\mathbf{r}_0$ at fixed $\mathcal{I}$, so that Eq. (2.11) can be effectively replaced by

$$\Gamma_R = 2\pi\eta^2 \sqrt{\frac{\epsilon_0}{3\nu}} \xi_R \int_{\mathcal{D}} d^2\mathbf{r}\, \tilde{\omega}(\mathbf{r})\sigma(\mathbf{r}), \quad (2.12)$$

where

$$\xi_R \equiv \frac{1}{\pi R^2} \int_{\mathcal{D}} d^2\mathbf{r}\, \xi(\mathbf{r}) = \frac{\xi_0}{\pi R^2} \int_{\mathcal{D}} d^2\mathbf{r} \sqrt{\frac{\epsilon(\mathbf{r})}{\epsilon_0}}. \quad (2.13)$$

In other words, in order to compute statistical properties of Γ_R, for $R \gg \eta_K$, we just need to deal with the much simpler random vorticity field

$$\omega_R(\mathbf{r}) = 2\pi\eta^2 \sqrt{\frac{\epsilon_0}{3\nu}} \xi_R \tilde{\omega}(\mathbf{r})\sigma(\mathbf{r}). \quad (2.14)$$

Incidentally, and fortunately, the very same expression as the above one is supposed to hold for the small-scale region $R \ll \eta_K$, since the dissipation field is not expected to exhibit fast spatial variations within dissipative length scales.

The spatially averaged field ξ_R has here a statistical role similar to the one of the coarse-grained three-dimensional dissipation field introduced in the OK62 phenomenology [13,14]. Recalling, in this connection, the statistically self-similar nature of multiplicative chaos processes [24], which model intermittent fluctuations of energy dissipation, we put forward that

$$\xi_R = \xi_0 \exp(-X_R) \quad (2.15)$$

holds for the present two-dimensional context, where X_R is a Gaussian random variable with both mean and variance given by

$$\bar{X}_R = \frac{3\mu}{8} \ln\left[\frac{R_\lambda}{\sqrt{15}}\left(\frac{\eta_K}{bR + \eta_K}\right)^{2/3}\right], \quad (2.16)$$

where R_λ is the Taylor-Reynolds number, $\mu = 0.17 \pm 0.01$ is the intermittency exponent [25], and $b > 0$ is a modeling parameter. The specific R-dependent expression between parentheses in (2.16) is proposed as an interpolation that works correctly for the $R/\eta_K \ll 1$ and $R/\eta_K \gg 1$ asymptotic cases. Note that (2.15) and (2.16) lead to $\langle \xi_R^4 \rangle \sim \langle \epsilon_R^2 \rangle \sim R^{-\mu}$ for $R \gg \eta_K$, as expected.

We are now ready to explore the model predictions given by Eqs. (1.1), (2.6)–(2.9), and (2.14)–(2.16), where the underlying $\xi(\mathbf{r})$, $\tilde{\omega}(\mathbf{r})$, and $\sigma(\mathbf{r})$ fields are assumed to be statistically independent from each other, on the basis of phenomenological expectations and theoretical constraints. Vortices are, to good approximation, advected by the turbulent velocity field, as prescribed by Kelvin's circulation theorem. Therefore, they are likely to be mixed in a chaotic way that renders their random spatial distribution, associated to $\sigma(\mathbf{r})$, to be independent (again, as a first approximation) from the circulations they carry. While the putative chaotic dynamics of vortex tubes still deserves deeper understanding, chaotic mixing is a well-established fact for two-dimensional point vortex systems [26]. In addition, the statistical independence of $\xi(\mathbf{r})$ and $\tilde{\omega}(\mathbf{r})$ in the definition of the vorticity field (2.4) is postulated mainly as a way to cope with two apparently contradictory statements: the fact, already discussed, that vorticity correlation functions in the plane γ should be factorized as a Gaussian stochastic process and the fact that the dissipation field, which is tied to the intensity of the circulation carried by vortex structures is non-Gaussian, usually modeled in terms of Gaussian multiplicative chaos [15].

III. STATISTICAL MOMENTS AND PROBABILITY DISTRIBUTION FUNCTIONS

Defining $\bar{R} = R/\eta_K$, the first interesting quantities to compute are the circulation variance and kurtosis in the asymptotic limit $\bar{R} \ll 1$. It turns out that up to lowest order in $\bar{R}$ dependence,

$$\langle \Gamma_R^2 \rangle = \frac{\epsilon_0}{3\nu}\xi_0^2 \bar{\sigma}\frac{\pi^3\eta^6}{a^4}\bar{R}^4, \quad (3.1)$$

$$\frac{\langle \Gamma_R^4 \rangle}{\langle \Gamma_R^2 \rangle^2} = \frac{3}{2}\frac{1}{\bar{\sigma}\pi\eta^2}\left(\frac{R_\lambda}{\sqrt{15}}\right)^{3\mu/2}\left(1 - \frac{1}{4a^2}\bar{R}^2\right). \quad (3.2)$$

Recalling that for $\bar{R} \ll 1$ one has $\langle \Gamma_R^2 \rangle \simeq \langle \omega^2 \rangle (\pi R^2)^2 = \epsilon_0(\pi R^2)^2/(3\nu)$, we find, from (3.1),

$$\xi_0^2 = \frac{1}{\bar{\sigma}\pi\eta^2}. \quad (3.3)$$

Also, from the data of the numerical simulations reported in [5] we obtain that $\lim_{\bar{R}\to 0}\langle \Gamma_R^4 \rangle/\langle \Gamma_R^2 \rangle^2 \simeq C_4 R_\lambda^{\alpha_4}$, where $C_4 \simeq 1.16$ and $\alpha_4 \simeq 0.41$. This implies, using (3.2), that

$$\bar{\sigma}\pi\eta^2 = \frac{3}{2}\frac{1}{C_4}\frac{1}{15^{3\mu/4}}R_\lambda^{(3\mu/2)-\alpha_4}. \quad (3.4)$$

Since $3\mu/2 - \alpha_4 = -0.14 < 0$, the above result tells us that the dilute vortex gas approximation improves the higher is the Reynolds number. For $R_\lambda = 240$, for instance, we already have $\bar{\sigma}\pi\eta^2 = 0.39$, meaning that the mean intervortex distance in this case can be estimated as $1/\sqrt{\bar{\sigma}} \simeq 2.8\eta$ and that vortex structures, which have core radius of the order of η, are in fact satisfactorily resolved.

Moving, now, to the opposite asymptotic region $\bar{R} \gg 1$, we get, keeping all the subdominant terms, the circulation kurtosis

$$\frac{\langle \Gamma_R^4 \rangle}{\langle \Gamma_R^2 \rangle^2} = \frac{\langle \xi_R^4 \rangle}{\langle \xi_R^2 \rangle^2}\frac{\sum_{n=1}^{6} A_n \bar{R}^{(4+2n)/3}}{\sum_{n=1}^{2} B_n \bar{R}^{(4+2n)/3}}, \quad (3.5)$$

where the exactly computed coefficients A_n's and B_n's are

$$\begin{aligned} &A_1 = 3\delta\pi + 6c^2\delta^2 G, \quad A_2 = 12c\delta^2 E,\\ &A_3 = 12c^2\delta^3 F, \quad A_4 = 3\delta^2\pi^2,\\ &A_5 = 6c\delta^3\pi E, \quad A_6 = 3c^2\delta^4 E^2,\\ &B_1 = \delta\pi, \quad B_2 = c\delta^2 E, \end{aligned} \quad (3.6)$$

with

$$\begin{aligned} &c = 1/[2\pi\Gamma(4/3)], \qquad \delta = \bar{\sigma}\eta_K^2,\\ &E = 4\pi^{5/2}\Gamma(5/6)\Gamma(2/3)/[\Gamma(4/3)\Gamma(7/3)],\\ &F = 0.33 \times (2\pi)^5, \qquad G = 2^{4/3}\pi^4\Gamma(2/3). \end{aligned} \quad (3.7)$$

It is worth emphasizing that b is the only free modeling parameter in Eq. (3.5). General moments of circulation can be computed for $\bar{R} \gg 1$, as well. At leading order, we find $\langle \Gamma_R^p \rangle \sim R^{4p/3}\langle \xi_R^p \rangle \sim R^{\lambda_p}$, where

$$\lambda_p = \frac{4p}{3} - \frac{\mu}{8}p(p-2). \quad (3.8)$$

The model predictions based on Eqs. (3.2), (3.5), and (3.8) lead to very suggestive comparisons to numerical results, as shown in Fig. 2. Equation (3.8) implies that $d\lambda_p/dp = 4/3 + \mu/4 = 1.376 \pm 0.002$ at $p = 0$. This result is in striking agreement with the value 1.367 ± 0.009 obtained from accurate numerical simulations [5].

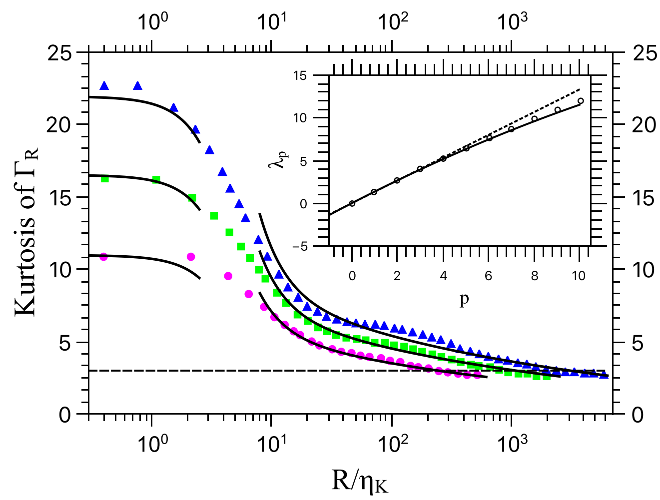

FIG. 2. Numerical kurtoses of circulation for $R_\lambda = 240$ (○), 650 (□), and 1300 (△) from [5]. Solid lines represent the asymptotic expansion, Eqs. (3.2), with $a = 3.3$, and (3.5), with $b = 2.0$, that hold at dissipative ($R \ll \eta_K$) and larger inertial range scales ($R \gg \eta_K$), respectively. The dashed line indicates, for reference, the kurtosis of a Gaussian distribution. The inset compares the numerical scaling exponents λ_p of circulation moments of order p evaluated at $R_\lambda = 1300$ [5] (open circles) with the predicted values computed from Eq. (3.8) (solid line). The straight dotted line gives the K41 scaling, $\lambda_p = 4p/3$.

The present approach, furthermore, allows us to derive closed analytical expressions for the cPDFs, $\rho_R(\Gamma)$, for $\bar{R} \gg 1$. Using (1.1) and (2.14), and representing, for convenience, the circulation in units of $2\pi\eta^2\sqrt{\epsilon_0/(3\nu)}$, the associated characteristic function is written as the triple expectation value

$$\begin{aligned} Z(\zeta) &= \langle\langle\langle \exp[i\zeta\Gamma_R]\rangle\rangle\rangle_{(\tilde{\omega},\xi_R,\sigma)}\\ &= \int_0^\infty d\xi\, f_R(\xi)\left\langle \exp\left[-\frac{1}{2}\zeta^2\xi^2\Omega\right]\right\rangle_\sigma, \end{aligned} \quad (3.9)$$

where $f_R(\xi)$ is the lognormal probability distribution function for the random variable ξ_R, defined from (2.15) and (2.16), and

$$\Omega \equiv \int_{\mathcal{D}} d^2\mathbf{r} \int_{\mathcal{D}} d^2\mathbf{r}' \langle \tilde{\omega}(\mathbf{r})\tilde{\omega}(\mathbf{r}')\rangle \sigma(\mathbf{r})\sigma(\mathbf{r}'). \quad (3.10)$$

A straightforward evaluation shows that the variance of Ω becomes very small compared to $\bar{\Omega}^2$ in the region $\bar{R} \gg 1$. Therefore, averaging over the $\sigma(\mathbf{r})$ fields is effectively equivalent to replacing Ω by its mean value $\bar{\Omega} \propto R^{8/3}$ in (3.9). Performing the Fourier transform of (3.9), we get

$$\rho_R(\Gamma) = \frac{1}{\sqrt{2\pi\bar{\Omega}}}\int_0^\infty d\xi \frac{1}{\xi} f_R(\xi)\exp\left(-\frac{\Gamma^2}{2\xi^2\bar{\Omega}}\right), \quad (3.11)$$

which is remarkably analogous to the Castaing *et al.* modeling form of velocity increment PDFs [28–30], with ξ taking the place of the energy dissipation rate ϵ. Additional analysis shows that for $1 \ll \bar{R} \ll R_\lambda^{3/2}$, the cPDF just obtained can be recast as a function of $\Gamma\exp(\bar{X}_R)/\sqrt{\bar{\Omega}} \sim \Gamma/R^h$ around

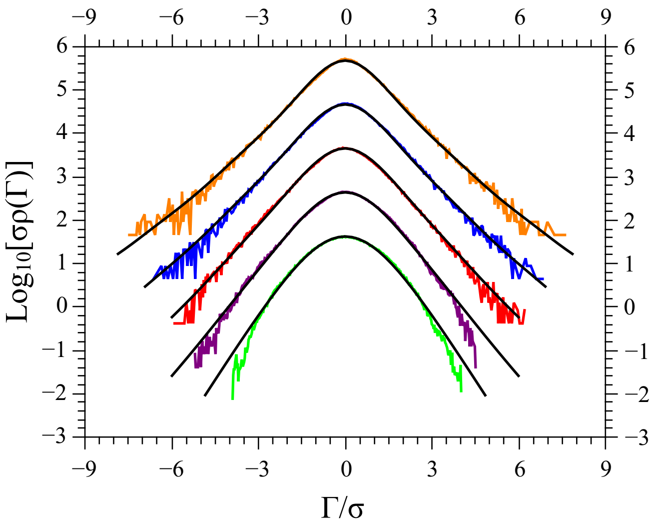

FIG. 3. Comparisons between standardized numerical cPDFs (vertically displaced to ease visualization) for $R/\eta_K = 16$, 32, 64, 128, and 256 (from the top to the bottom) obtained at $Re_\lambda = 418$ from the Johns Hopkins University turbulence database [27], and the ones evaluated from the theoretical result (solid black lines) given by Eq. (3.11). The cPDF tails are in general slightly underestimated due to finite-size ensemble effects, with more pronounced drops taking place at larger circulation contours.

its central core, where $h = 4/3 + \mu/4 = 1.376$, in agreement with the observations of Ref. [5]. Numerical cPDFs computed from the Johns Hopkins University turbulence data base [27] are closely reproduced by Eq. (3.11), as indicated in Fig. 3.

IV. CONCLUSIONS

To summarize, we have been able to address several important statistical features of turbulent circulation, relying on the fusion of structural concepts—the picture of a turbulent flow as a system of sparse vortex tubes—with the long known OK62 phenomenological approach to intermittency. The vortex gas modeling introduced here throws light on the dependence of the circulation kurtosis with the Reynolds number and probing scale, the scaling exponents of the circulation moments, the detailed shape of cPDFs, and their related collapsing exponent $h \simeq 1.4$.

A few comments on the domain of validity of the results reported in Fig. 2 are in order. As was shown in Ref. [5], the scaling exponents of circulation moments are best fitted by a linear function of the moment order p for $p > 6$. In contrast, our modeling relies heavily on the structure of OK62 phenomenology, where intermittency scaling exponents are necessarily quadratic functions of the moment orders.

The linearization of scaling exponents at high moment orders is not a particular feature of circulation statistics, but it is also observed in the evaluation of velocity structure functions. It is known, for the latter, that the concavity problems [11] associated to OK62 phenomenology can be fixed within the multifractal formalism, assuming that the multifractal set of field singularities degenerates into a monofractal set characterized by a minimum Hölder exponent [31,32]. We remark that an analogous discussion could be applied, in principle, to model the scaling properties of circulation moments.

Further progress is likely to be more conveniently approached by Monte Carlo simulations of the fields $\tilde{\omega}(\mathbf{r})$, $\xi(\mathbf{r})$, and $\sigma(\mathbf{r})$, under a variety of model definitions. A problem of fundamental interest, of course, is how to extend (2.4) or (2.14) to the case of nonplanar loops, taking into account possible connections of circulation statistics with minimal surface theory [6,33]. The formulation of a bridge between the languages of vortex gas modeling and multifractality is also an exciting problem, which surely deserves close attention in future investigations.

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