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Integral Transforms with Single Domain Formulation for Transient Three-Dimensional Conjugated Heat Transfer

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ABSTRACT

An analysis based on integral transforms is undertaken for transient three-dimensional conjugated conduction-convection heat transfer, with a focus on mini or micro channels-based devices. The numerical-analytical approach Generalized Integral Transform Technique (GITT) is combined with a single domain reformulation, providing accurate, robust, and cost-effective simulations for determining temperature distributions within the domain. The fluid and solid subdomains are represented as one single region, while the integral transformation is carried out using a three-dimensional eigenvalue problem encompassing the thermophysical properties and velocity field abrupt spatial variations. The steady state problem solution is employed as a filter and solved through an integral transformation based on the corresponding two-dimensional eigenvalue problem defined for the channel cross section. The transformed ordinary differential systems for both the steady and homogeneous transient problems are handled analytically, requiring only the numerical solution of the associated matrix eigensystem analysis. Converged numerical results for dimensionless temperature distributions are then critically compared with a finite element solution and a previously proposed GITT solution that implements a partial transformation scheme under a pseudo-transient formulation, both for the conjugated problem with thermally developing laminar flow in a rectangular channel.

Introduction

Conjugated conduction-convection heat transfer is one of the most classical problem formulations in thermal sciences. While an exact solution is not yet available to the general situation, a few approximate analytical solutions have been proposed throughout the last few decades, since the pioneering works of Perelman [1] and Luikov et al. [2], especially for two-dimensional steady formulations, either as fully differential or mixed lumped-differential models. Applications are numerous, in both the physical sciences and engineering contexts, appearing whenever solid structures or flow inserts cannot be disregarded when analyzing the heat transfer process in the whole fluid-solid thermal system. Then, classical correlations or analytical solutions for purely convective heat transfer are no longer applicable, and computational fluid mechanics tools are forcedly required to provide costly numerical solutions to such inherently conjugated problems.

Microfluidics is a fairly recent scientific domain characterized by fluid flow and heat and mass transfer at the micro-scale and the associated applications [3–5]. A literature survey on Google Scholar with the keyword "microfluidics", which includes journal papers, conferences and patents, results in an ever-increasing number of documents since 1981, with a marked increase after the years 2000, and more than 20,000 documents per year in 2020 [6]. Process intensification is the main drive behind this miniaturization wave of various classes of devices, including micro-mixers, micro-pumps, micro-separators, micro-reactors, microfluidic batteries, and micro-heat exchangers, among others. An increasing focus in the research on heat and mass transfer at the micro-scale, with focus on such different micro-devices, has been evident in the open literature in

Nomenclature

A	Area of the cross section of the channel, m^2
AR	Aspect ratio
c	Coefficients of the solution of the first order transformed problem
c_p	Specific heat, J/kgK
D_h	Hydraulic diameter, m
GITT	Generalized Integral Transform Technique
ITRS	International Technology Roadmap for Semiconductors
k	Thermal conductivity, W/mK
K	Dimensionless thermal conductivity
$\mathbf{L}$	Coefficients matrix for the first order transformed problem
L_{x_T}	Microfluidic apparatus width in x direction, m
L_{y_T}	Microfluidic apparatus width in y direction, m
L_{x_M}	Microchannel width in x direction, m
L_{y_M}	Microchannel width in y direction, m
N	Normalization integrals
ODE	Ordinary Differential Equation
PDMS	Polydimethylsiloxane
Pe	Péclet number
t	Time variable, s
T	Temperature, °C
$\mathbf{u}$	Velocity vector, m/s
u	Velocity component along the z-direction, m/s
U	Dimensionless velocity component along the z-direction
w	Volumetric heat capacity (ρc_p), J/m^{3}K
W	Dimensionless volumetric heat capacity
x, y, z	Coordinates, m
X, Y, Z	Dimensionless coordinates
z_∞	Length of the device, m
Z_∞	Dimensionless length of the device

Greek symbols

α	Thermal diffusivity, m^2/s
α_j	Eigenvalues of matrix $\mathbf{L}$
β, ν, μ, λ	Eigenvalues associated to eigenfunctions ξ, Ω, Ψ, χ, respectively
Γ, ω	Auxiliary eigenfunctions
$\delta_{i,j}$	Kronecker's delta
$\boldsymbol{\zeta}$	Matrix of eigenvectors of $\mathbf{L}$
ζ	Eigenvectors of $\mathbf{L}$
θ	Dimensionless temperature profile
ξ, Ω	Eigenfunctions of the steady conjugated problem
ρ	Density, kg/m^3
σ_x	Ratio between the hydraulic diameter and the width of the device
σ_y	Ratio between the hydraulic diameter and the height of the device
τ	Dimensionless time
$\boldsymbol{\varphi}$	Vector of dependent functions
ψ, χ	Eigenfunctions of the homogeneous transient conjugated problem

Subscripts

av	Related to the average field
f	Fluid region
H	Related to the homogeneous transient problem
i, k, m, n	Indices associated with the eigenvalue problems
in	Related to the inlet conditions
P	Related to the steady state problem
s	Solid region
w	Related to the channel wall
x	Refers to the x-direction
y	Refers to the y-direction

Superscripts

$\sim$	Normalized eigenfunction
$-$	Transformed potential

recent decades, driven by the ever-growing demand for lighter, smaller and more efficient micro-systems [5, 7–10]. For instance, since the pioneering work of Tuckerman and Pease [11], the reduced size of the channels and the associated rise in heat dissipation capacity, rendered micro-heat exchangers an interesting option for the cooling of electronic devices. In 2012, the *International Technology Roadmap for Semiconductors* (ITRS) anticipated that, in 2020, the dissipated power of integrated circuits would be above 100 W/cm^2 [12]. Since then, three-dimensional integrated circuits have been advanced, outpacing ITRS roadmap forecasts, while requiring cooling based on an interlayer microchannels structure [13–15]. In addition, single phase microchannels-based heat exchangers have been the preferred option in several heat removal intensive applications, ranging from high concentration photovoltaics, 3D chip stacks, fuel cells, micro-reactors, among others [12, 16–23].

Along the development of micro-devices that involve heat transfer, deviations have been observed between experimental and theoretical results of important performance parameters [24, 25], which have been diminishing in recent years, largely on account of advances in measurement techniques. Nonetheless, discrepancies were also observed while adopting correlations originally derived from macroscale experiments to verify quantities in microchannels measurements, pointing out to the necessity of proposing specific correlations for microscale devices. One of the claims is that such deviations can be mainly attributed to disregarding phenomena that, differently from the macroscale, may have significant importance in a thermal microsystem. These effects are known as scaling effects, such as wall conjugation, viscous dissipation, entry effects, measurement uncertainty, axial heat diffusion, slip and temperature jump, compressibility, among others [10, 24, 26–33]. There

is still a disagreement in the literature regarding which scaling effects are essential on the correct treatment of heat transfer problems at the microscale, for each specific application, but there is strong evidence that the conjugation effect is highly relevant and its influence must be, at least, checked for in every case [10, 27, 28, 31]. Thus, a better agreement between experimental and simulation results for convective heat transfer in microchannels requires research on models and solution methodologies that are capable of adequately describing the relevant physical phenomena to fluid flow and heat transfer at such scales.

Not rarely, the solid substrates and the channels of thermal micro devices are of comparable sizes, and, besides, due to combinations of physical dimensions and properties, in general participate in the overall heat transfer process. Thus, for the theoretical analysis, design and optimization of micro-devices that involve heat transfer, it is commonly mandatory to study the conjugated heat transfer problem. As previously mentioned, since the general class of conjugated conduction-convection problems in full partial differential formulation does not have known exact analytical solutions, purely numerical or hybrid (numerical-analytical) methods are possible alternatives. However, due to the presence of multiple geometric scales and abrupt interfaces between distinct materials, purely numerical approaches in general require refined discretization meshes in different sub-regions and their transitions, which results in considerable computational costs. In addition, purely numerical schemes suffer from soaring computational costs when dealing with three-dimensional formulations. Therefore, previous works have dealt with hybrid numerical-analytical solutions for this class of conjugated problems, offering robustness, accuracy and, in particular, mild computational effort. This aspect becomes more evident in highly intensive computational tasks, such as in optimization, inverse problems, and simulation under uncertainty, when a very large number of runs in the direct problem solution is required. One such hybrid approach is known as the Generalized Integral Transform Technique - GITT [34–43] which derives from the Classical Integral Transform Method, Mikhailov and Ozisik [44]. The GITT offers solutions with robustness and error control similar to its purely analytical version, significantly expanding the coverage of this more flexible hybrid method in different classes of problems that are, in principle, non-transformable in the analytical sense. The GITT solution of conjugated problems was first proposed in the realm of periodic convection

with conjugation effects in the form of just a transversal thermal resistance but considering the wall thermal capacitance [45, 46]. Soon afterwards, the axial conduction along the wall was considered, but still simplifying the formulation with respect to the transversal wall conduction through a partial lumped analysis [47–49]. This same concept was later extended through an improved lumped-differential analysis [39, 50], being applied and experimentally validated in the analysis of rectangular micro-heat exchangers [51]. The proposition of a single domain reformulation strategy allowed for the solution of full conjugated heat transfer problems through the introduction of spatially variable coefficients without any approximation to the wall heat conduction process [52]. The approach was then extended to include other scaling effects and generalizing the formulation for regular and complex geometries in steady or transient regimen [53–60]. More recently, Knupp et al. [61] developed a convective eigenvalue problem and achieved a convergence acceleration in solving conjugated problems for parallel plate channels via GITT.

Besides extending the integral transform approach to progressively more involved conjugated problem formulations, a few works have previously addressed transient conjugated problems, including periodic disturbances or fluctuations in the flow field, and/or wall and entry conditions, which only under very special circumstances can be reduced to quasi-steady formulations [62–66]. Though a two-dimensional transient formulation [56] and a three-dimensional steady formulation [55] have been previously dealt with GITT, a total transformation scheme for transient three-dimensional conjugated problems has not yet been implemented. As for the full transient formulation in micro heat exchangers, although the reduced spatial scales lead *a priori* to very small-time scales in the convective heat transfer transients, it should be recalled that the substrate thermal capacitance may play a major role in the time evolution of flow or temperature disturbances, requiring the full transient analysis of the thermal problem.

In this context, the present work aims to contribute with a hybrid numerical-analytical solution of a full transient three-dimensional conjugated problem formulation in thermal microsystems, through the combined use of GITT and the single domain reformulation strategy. In microchannels, Reynolds numbers are typically low and characteristic of laminar flows. Also, the hydrodynamic development lengths are in general very small relative to the total length of the exchanger, and the flow entrance effects

are negligible, leading to a thermally developing flow situation. On the other hand, Peclet number may be sufficiently low to require the inclusion of the fluid axial diffusion term in the energy equation. The focus of the present work is the generalization of the hybrid methodology for the treatment, via total transformation, of the transient three-dimensional energy equation, written for both the solid and fluid domains. Following the single domain strategy, the problem is reformulated through unified spatially variable coefficients that include the abrupt spatial variations associated with the properties and velocities of fluid and solid. Thus, the integral transform method solves one single energy equation for the entire region, obtaining the transient temperature field through the expansion of the appropriate three-dimensional eigenfunctions. The spatial variation of the coefficients from the original problem is fully represented within the eigenvalue problem, which is itself solved via GITT due to its inherent complexity and to the inexistence of an analytical solution.

Problem formulation

Consider incompressible laminar flow of a Newtonian fluid inside a rectangular microchannel, with dimensions and coordinates system defined as in Figure 1, and external walls subjected to a prescribed uniform temperature T_w, different from the system initial temperature, T_{in}. Two scaling effects are here considered, namely the wall conjugation and the axial heat diffusion effects. The channel walls participate in the heat transfer process both in the transversal and longitudinal directions. The flow is considered hydrodynamically developed and thermally developing from a uniform inlet temperature, T_{in}, the same as the initial temperature of the whole device, with temperature independent physical properties. The hydrodynamically developed flow assumption is associated with the

presence of an upstream adiabatic channel section where the fully developed flow profile is achieved. Then, the transient three-dimensional conjugated problem is reformulated as a single domain energy equation as:

$$w(x,y)\left[\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T\right] = \nabla \cdot \left[k(x,y)\nabla T\right], 0 < x < L_{x_T},$$
$$0 < y < L_{y_T}, 0 < z < z_\infty, t > 0 \quad \text{(1a)}$$

$$T(0,y,z,t) = T_w; T(L_{x_T},y,z,t) = T_w;$$
$$0 \le y \le L_{y_T}, 0 \le z \le z_\infty, t > 0 \quad \text{(1b,c)}$$

$$T(x,0,z,t) = T_w; T(x,L_{y_T},z,t) = T_w;$$
$$0 \le x \le L_{x_T}, 0 \le z \le z_\infty, t > 0 \quad \text{(1d,e)}$$

$$T(x,y,0,t) = T_{in}; \left.\frac{\partial T}{\partial z}\right|_{z=z_\infty} = 0;$$
$$0 \le x \le L_{x_T}, 0 \le y \le L_{y_T} \quad \text{(1f,g)}$$

$$T(x,y,z,0) = T_{in}; 0 \le x \le L_{x_T}, 0 \le y \le L_{y_T}, 0 \le z \le z_\infty \quad \text{(1h)}$$

The thermophysical properties and the fully developed velocity profile are incorporated into spatially variable coefficients, so as to represent the abrupt transitions between the fluid and solid regions as:

$$k(x,y) = \begin{cases} k_f, & \text{at fluid region} \\ k_s, & \text{at substrate region} \end{cases} \quad \text{(2a)}$$

$$w(x,y) = \begin{cases} w_f, & \text{at fluid region} \\ w_s, & \text{at substrate region} \end{cases} \quad \text{(2b)}$$

$$u(x,y) = \begin{cases} u_f(x,y), & \text{at fluid region} \\ 0, & \text{at substrate region} \end{cases} \quad \text{(2c)}$$

where w_f and w_s are, respectively, the fluid and solid volumetric heat capacities, whereas k_f and k_s are, respectively, the fluid and solid thermal conductivities, and $u_f(x,y)$ is the fully developed velocity
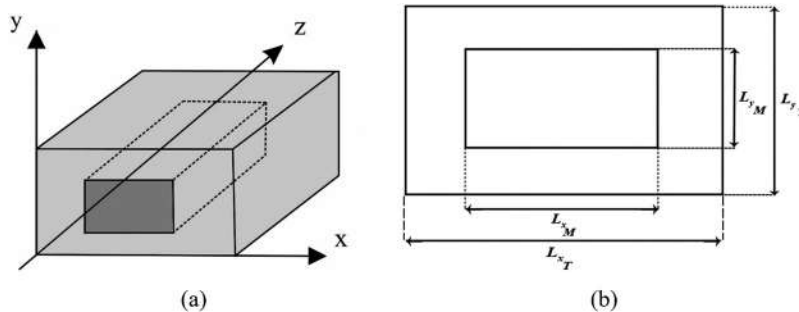

Figure 1. Schematic representation of the transient three-dimensional conjugated problem in a rectangular microchannel and corresponding solid substrate. (a) Coordinates system and (b) Definitions for the height and width of the substrate and microchannel.

profile in the fluid. The dimensionless groups are defined as:

$$W = \frac{w}{w_f}; \theta = \frac{T - T_w}{T_{in} - T_w}; \tau = \frac{\alpha_f t}{D_h^2} \qquad (3\text{a–c})$$

$$X = \frac{2x}{L_{x_T}}; Y = \frac{2y}{L_{y_T}}; Z = \frac{z}{D_h Pe} \qquad (3\text{d–f})$$

$$U = \frac{u}{u_{av}}; K = \frac{k}{k_f}; Pe = \frac{u_{av} D_h}{\alpha_f}; \alpha_f = \frac{k_f}{w_f}; D_h = 2\frac{L_{x_M} L_{y_M}}{L_{x_M} + L_{y_M}} \qquad (3\text{g–k})$$

$$\sigma_x = \frac{L_{x_T}}{D_h}; \sigma_y = \frac{L_{y_T}}{D_h}; X_i = \frac{L_{x_M}}{L_{x_T}}; Y_i = \frac{L_{y_M}}{L_{y_T}}; AR = \frac{Y_i}{X_i} \qquad (3\text{l–p})$$

where u_{av} is the average flow velocity. The dimensionless transversal space variables were taken from 0 to 2 to match the definitions in Ref. [55] for comparison purposes, which employed the limits from 0 to 1, but imposed symmetry of the boundary conditions on the transversal directions. Here, symmetry conditions were not imposed, so as to generalize the approach for non-symmetric boundary conditions and/or channel geometries. Equations (1a–h) can then be rewritten in dimensionless form as:

$$W(X,Y)\left[\frac{\partial\theta}{\partial\tau} + U(X,Y)\frac{\partial\theta}{\partial Z}\right] = \frac{4}{\sigma_x^2}\frac{\partial}{\partial X}\left[K(X,Y)\frac{\partial\theta}{\partial X}\right]$$
$$+ \frac{4}{\sigma_y^2}\frac{\partial}{\partial Y}\left[K(X,Y)\frac{\partial\theta}{\partial Y}\right] + \frac{1}{Pe^2}\frac{\partial}{\partial Z}\left[K(X,Y)\frac{\partial\theta}{\partial Z}\right],$$
$$0 < X < 2, \ 0 < Y < 2, \ 0 < Z < Z_\infty, \ \tau > 0 \qquad (4\text{a})$$

$$\theta(0,Y,Z,\tau)=0; \theta(2,Y,Z,\tau)=0; 0\le Y\le 2, 0\le Z\le Z_\infty, \tau>0 \qquad (4\text{b,c})$$

$$\theta(X,0,Z,\tau)=0; \theta(X,2,Z,\tau)=0; 0\le X\le 2, 0\le Z\le Z_\infty, \tau>0 \qquad (4\text{d,e})$$

$$\theta(X,Y,0,\tau)=1; \frac{\partial\theta}{\partial Z}\bigg|_{Z=Z_\infty}=0; 0\le X\le 2, 0\le Y\le 2, \tau>0 \qquad (4\text{f,g})$$

$$\theta(X,Y,Z,0)=1; 0\le X\le 2, 0\le Y\le 2, 0\le Z\le Z_\infty \qquad (4\text{h})$$

The dimensionless thermophysical properties and velocity profile are defined, for a straight rectangular channel, as:

$$K(X,Y) = \begin{cases} 1, 1-X_i \le X \le 1+X_i, 1-Y_i \le Y \le 1+Y_i \\ k_f/k_s, \qquad\qquad otherwise \end{cases} \qquad (5\text{a})$$

$$W(X,Y) = \begin{cases} 1, 1-X_i \le X \le 1+X_i, 1-Y_i \le Y \le 1+Y_i \\ w_f/w_s, \qquad\qquad otherwise \end{cases} \qquad (5\text{b})$$

$$U(X,Y) = \begin{cases} \dfrac{u_f(x,y)}{u_{av}}, 1-X_i \le X \le 1+X_i, 1-Y_i \le Y \le 1+Y_i \\ 0, \qquad\qquad otherwise \end{cases} \qquad (5\text{c})$$

The dimensionless velocity profile is obtained for fully developed flow in a rectangular channel, which has analytical solution by separation of variables, as presented in Ref. [36]:

$$\frac{u_f(x,y)}{u_{av}} = U_f(X,Y) = A^* \sum_{j=1,3,5,\ldots}^{\infty} B_j F_j(Y) G_j(X) \qquad (6\text{a})$$

$$A^* = \frac{48}{\pi^3}\left[1 - \frac{192}{\pi^5}\frac{X_i}{Y_i}\sum_{l=1,3,5,\ldots}^{\infty}\frac{1}{l^5}\tanh\left(\frac{l\pi Y_i}{2X_i}\right)\right]^{-1};$$
$$B_j = \frac{(-1)^{(j-1)/2}}{j^3} \qquad (6\text{b,c})$$

$$F_j(Y) = 1 - \frac{\cosh(a_j Y)}{\cosh(a_j Y_i)}; G_j(X) = \cos(a_j X); a_j = \frac{j\pi}{2X_i} \qquad (6\text{d–f})$$

Solution methodology

Integral transform solutions for transient conjugated heat transfer in two-dimensional parallel plate channels [56] and for steady-state three-dimensional rectangular channels [54] have been previously obtained. In both works, the partial transformation scheme of the GITT was employed, constructing eigenfunction expansions in the transversal coordinates only, and leaving the axial variable untransformed. Then, a transformed partial differential system results, having the longitudinal coordinate and time (for the transient problem or a pseudo-transient formulation for the steady problem) as independent variables, which is numerically handled to yield the transformed temperatures. The partial transformation path reduces the analytical involvement and, in many cases, competes on computational cost with the more traditional total transformation scheme. Nevertheless, accuracy requirements might favor the adoption of the total transformation alternative, eliminating all the space variables in analytic form and solving an ordinary differential transformed system instead. In addition, for a linear transformed ordinary differential equation

(ODE) system that is analytically solvable, there might be a significant gain in overall computational cost as well. Thus, a total integral transformation scheme combined with the single domain reformulation strategy is here employed in solving transient three-dimensional conjugated problems.

For improved convergence of the eigenfunction expansion, it is always recommended to reduce the importance of the equation and boundary source terms, with the associated gain in computational performance [36]. Therefore, the following filtering solution for problem (4) is proposed:

$$\theta(X, Y, Z, \tau) = \theta_H(X, Y, Z, \tau) + \theta_P(X, Y, Z) \quad (7)$$

where $\theta_P(X, Y, Z)$ is the solution of the steady-state conjugated problem, which is here defined as a filtering solution, and $\theta_H(X, Y, Z, \tau)$ is the solution of the homogeneous version of the transient problem (4).

Steady state conjugated problem

With respect to the present transient problem solution, other simpler filters can be proposed, but the steady state solution is well known for having excellent characteristics for convergence acceleration of transient problems, and for this reason it has been the adopted filter. The steady state solution could have been obtained by the pseudo-transient solution path in the partial transformation scheme [55], but here a total double transformation is preferred, reaching a transformed ODEs system in the longitudinal coordinate only. The dimensionless steady conjugated problem that defines the filter is given by:

$$W(X, Y)U(X, Y)\frac{\partial \theta_P}{\partial Z} = \frac{4}{\sigma_x^2}\frac{\partial}{\partial X}\left[K(X, Y)\frac{\partial \theta_P}{\partial X}\right]$$
$$+ \frac{4}{\sigma_y^2}\frac{\partial}{\partial Y}\left[K(X, Y)\frac{\partial \theta_P}{\partial Y}\right] + \frac{K(X, Y)}{Pe^2}\frac{\partial^2 \theta_P}{\partial Z^2}$$

$$(8a)$$

$$\theta_P(0, Y, Z) = 0; \theta_P(2, Y, Z) = 0; 0 \le Y \le 2, 0 \le Z \le Z_\infty$$
$$(8b,c)$$

$$\theta_P(X, 0, Z) = 0; \theta_P(X, 2, Z) = 0; 0 \le X \le 2, 0 \le Z \le Z_\infty$$
$$(8d,e)$$

$$\theta_P(X, Y, 0) = 1; \frac{\partial \theta_P}{\partial Z}\bigg|_{Z=Z_\infty} = 0; 0 \le X \le 2, 0 \le Y \le 2$$

$$(8f,g)$$

The chosen transformation strategy for solving Eq. (8) is based on a two-dimensional eigenfunction expansion to eliminate the transversal coordinates, leading to a transformed boundary value problem

along the longitudinal coordinate, Z. The following integral transform pair is then defined:

Transform:

$$\overline{\theta}_{P_i}(Z) = \int_0^2 \int_0^2 \frac{K(X, Y)}{Pe^2}\theta_P(X, Y, Z)\tilde{\xi}_i(X, Y)dXdY$$

$$(9a)$$

Inverse:

$$\theta_P(X, Y, Z) = \sum_{i=1}^\infty \tilde{\xi}_i(X, Y)\overline{\theta}_{P_i}(Z) \quad (9b)$$

The normalized eigenfunction and its associated eigenvalues come from the following proposed eigenvalue problem:

$$\frac{4}{\sigma_x^2}\frac{\partial}{\partial X}\left[K(X, Y)\frac{\partial \xi_i}{\partial X}\right] + \frac{4}{\sigma_y^2}\frac{\partial}{\partial Y}\left[K(X, Y)\frac{\partial \xi_i}{\partial Y}\right]$$
$$+ \beta_i^2 \frac{K(X, Y)}{Pe^2}\xi_i(X, Y) = 0$$

$$(10a)$$

$$\xi_i(0, Y) = 0; \xi_i(2, Y) = 0, 0 \le Y \le 2 \quad (10b,c)$$

$$\xi_i(X, 0) = 0; \xi_i(X, 2) = 0, 0 \le X \le 2 \quad (10d,e)$$

and the normalized eigenfunctions are calculated from:

$$\tilde{\xi}_i(X, Y) = \frac{\xi_i(X, Y)}{\sqrt{N_{\xi_i}}} \quad (10f)$$

with the normalization integrals given by:

$$N_{\xi_i} = \int_0^2 \int_0^2 \frac{K(X, Y)}{Pe^2}\xi_i(X, Y)^2 dXdY \quad (10g)$$

This eigenvalue problem with space variable coefficients can be handled by the GITT itself with the proposition of a simpler auxiliary problem, expanding the unknown eigenfunctions in terms of the chosen basis, following the procedure described in Refs. [10, 42]. Thus, a simpler auxiliary problem is defined as:

$$\frac{4}{\sigma_x^2}\frac{\partial^2 \Omega_k}{\partial X^2} + \frac{4}{\sigma_y^2}\frac{\partial^2 \Omega_k}{\partial Y^2} + \nu_k^2 \Omega_k(X, Y) = 0, 0 < X < 2, 0 < Y < 2$$

$$(11a)$$

$$\Omega_k(0, Y) = 0; \Omega_k(2, Y) = 0, 0 \le Y \le 2 \quad (11b,c)$$

$$\Omega_k(X, 0) = 0; \Omega_k(X, 2) = 0, 0 \le X \le 2 \quad (11d,e)$$

Applying separation of variables to problem (11), it is found that the auxiliary eigenfunction can be expressed as a product of two one-dimensional eigenfunctions, as:

$$\Omega(X, Y) = \omega(X)\Gamma(Y) \quad (12a)$$

where,

$$\frac{4}{\sigma_x^2}\frac{d^2\omega}{dX^2} + \nu_x^2\omega(X) = 0 \qquad (12b)$$

$$\omega(0) = 0; \omega(2) = 0 \qquad (12c,d)$$

$$\frac{4}{\sigma_y^2}\frac{d^2\Gamma}{dY^2} + \nu_y^2\Gamma(Y) = 0 \qquad (12e)$$

$$\Gamma(0) = 0; \Gamma(2) = 0 \qquad (12f,g)$$

Problems (12) are readily solved in analytical form and the auxiliary eigenfunctions and eigenvalues are given by:

$$\Omega_k(X, Y) = \Omega_{m,n}(X, Y) = \sin\left(\frac{m\pi X}{2}\right)\sin\left(\frac{n\pi Y}{2}\right),$$

$$\text{with} \quad \nu_k^2 = \nu_{m,n}^2 = \nu_{x_m}^2 + \nu_{y_n}^2 \qquad (13a,b)$$

Then, the following transform-inverse pair is proposed for the expansion of the original eigenfunctions, with the corresponding normalized eigenfunction and normalization integral:

Transform:

$$\overline{\xi}_{m,n}^{(i)} = \int_0^2\int_0^2 \tilde{\Omega}_{m,n}(X, Y)\xi_i(X, Y)dXdY \qquad (14a)$$

Inverse:

$$\xi_i(X, Y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\tilde{\Omega}_{m,n}(X, Y)\overline{\xi}_{m,n}^{(i)} \qquad (14b)$$

$$\tilde{\Omega}_{m,n}(X, Y) = \frac{\Omega_{m,n}(X, Y)}{\sqrt{N_{\Omega_{m,n}}}}, N_{\Omega_{m,n}} = \int_0^2\int_0^2\Omega_{m,n}(X, Y)^2 dXdY \qquad (14c,d)$$

For computational purposes, it is preferable to rewrite the double summation of the inverse formula (14b) as a single summation according to an appropriate reordering rule, to account for the most relevant terms of the multiple summation in the final numerical result [39]. Since the final solution is not known a priori, the simplest proposition is the ascending order of the squared eigenvalues $\nu_{m,n}^2$ defined in Eq. (13b). Thus, for each index k is associated a pair (m, n) corresponding to the indices in the eigenvalues for each coordinate direction, as given by Eqs. (13a,b), once the eigenvalues are reorganized in ascending order [39]. Then, the inverse formula (14b) becomes:

$$\xi_i(X, Y) = \sum_{k=1}^{\infty}\tilde{\Omega}_k(X, Y)\overline{\xi}_k^{(i)} \qquad (14e)$$

The integral transformation of the eigenvalue problem with space variable coefficients is then achieved by operating with $\int_0^2\int_0^2\tilde{\Omega}_k(X, Y) - dXdY$, and substituting

the inverse formula (14e), to obtain the following algebraic eigensystem:

$$(\mathbf{A} + \mathbf{C})\overline{\xi} = \beta^2\mathbf{B}\overline{\xi} \qquad (15a)$$

$$A_{kl} = -\int_0^2\int_0^2\left\{\frac{4}{\sigma_x^2}[1 - K(X, Y)]\frac{\partial\tilde{\Omega}_k}{\partial X}\frac{\partial\tilde{\Omega}_l}{\partial X} \right. \\ \left. + \frac{4}{\sigma_y^2}[1 - K(X, Y)]\frac{\partial\tilde{\Omega}_k}{\partial Y}\frac{\partial\tilde{\Omega}_l}{\partial Y}\right\}dXdY \qquad (15b)$$

$$C_{kl} = \nu_k^2\delta_{kl}; B_{kl} = \int_0^2\int_0^2\frac{K(X, Y)}{Pe^2}\tilde{\Omega}_k(X, Y)\tilde{\Omega}_l(X, Y)dXdY \qquad (15c,d)$$

Well-established algorithms for matrix eigensystem analysis can readily solve the algebraic problem given by Eq. (15), obtaining the vector of eigenvalues β and the eigenvectors $\overline{\xi}$, after truncation to a sufficiently large finite order N_F. Then, operating the steady energy equation (8a) with $\int_0^2\int_0^2\tilde{\xi}_i(X, Y) - dXdY$ and making use of the inverse formula and boundary conditions, it results the following coupled system of ordinary differential equations for the steady state transformed temperatures:

$$\sum_{j=1}^{\infty}M_{ij}\frac{d\overline{\theta}_{P_j}}{dZ} = -\beta_i^2\overline{\theta}_{P_i}(Z) + \frac{d^2\overline{\theta}_{P_i}}{dZ^2} \qquad (16a)$$

$$\overline{\theta}_{P_i}(0) = \int_0^2\int_0^2\frac{K(X, Y)}{Pe^2}\tilde{\xi}_i(X, Y)dXdY; \frac{d\overline{\theta}_{P_i}}{dZ}\Big|_{Z=Z_\infty} = 0 \qquad (16b,c)$$

$$M_{ij} = \int_0^2\int_0^2 W(X, Y)U(X, Y)\tilde{\xi}_i(X, Y)\tilde{\xi}_j(X, Y)dXdY \qquad (16d)$$

The infinite system is then truncated to a sufficiently large order N_P and, despite being a coupled system, may be solved analytically via diagonalization of the coefficients matrix in the ODE system. Rewriting Eqs. (16a–d) as a first order ODE system, we then have:

$$\overline{\theta}_{P_{i+N_P}}(Z) = \frac{d\overline{\theta}_{P_i}}{dZ}; \sum_{j=1}^{N_P}M_{ij}\overline{\theta}_{P_{j+N_P}}(Z) = -\beta_i^2\overline{\theta}_{P_i}(Z) + \frac{d\overline{\theta}_{P_{i+N_P}}}{dZ} \qquad (16e,f)$$

$$\overline{\theta}_{P_i}(0) = \int_0^2\int_0^2\frac{K(X, Y)}{Pe^2}\tilde{\xi}_i(X, Y)dXdY; \frac{d\overline{\theta}_{P_i}}{dZ}\Big|_{Z=Z_\infty} = 0 \qquad (16g,h)$$

or in matrix form,

$$\boldsymbol{\varphi}' + \mathbf{L}\boldsymbol{\varphi} = 0 \qquad (17a)$$

where $\mathbf{L}$ is the coefficients matrix of order $2N_P$, independent of the axial coordinate, and the solution vector is formed by:

$$\boldsymbol{\varphi} = \left\{ \overline{\theta}_{P_1}(Z), ..., \overline{\theta}_{P_{N_P}}(Z), \frac{d\overline{\theta}_{P_{N_P+1}}}{dZ}, ..., \frac{d\overline{\theta}_{P_{2N_P}}}{dZ} \right\}^T \quad (17b)$$

The corresponding matrix eigensystem problem for solving Eq. (17a) is then given by:

$$(\mathbf{L} - \alpha \mathbf{I})\boldsymbol{\zeta} = 0 \quad (18)$$

where $\mathbf{I}$ is the identity matrix, α are the eigenvalues of the matrix $\mathbf{L}$, and $\boldsymbol{\zeta}$ are the associated eigenvectors. The solution for the transformed potentials and their derivatives is then constructed as:

$$\varphi_i(Z) = \sum_{j=1}^{2N_P} c_j \zeta_{i,j} e^{-\alpha_j Z}, \quad i = 1, ..., 2N_P \quad (19a)$$

where the constants c_j are obtained so as to satisfy the boundary conditions, yielding the following linear algebraic system:

$$c_1 \zeta_{i,1} + ... + c_{2N_P} \zeta_{i,2N_P} = \int_0^2 \int_0^2 \frac{K(X,Y)}{Pe^2} \tilde{\xi}_i(X,Y) dX dY, i = 1, ..., N_P \quad (19b)$$

$$c_1 \zeta_{i,1} e^{-\alpha_1 Z_\infty} + ... + c_{2N_P} \zeta_{i,2N_P} e^{-\alpha_{2N_P} Z_\infty} = 0, i = N_P + 1, ..., 2N_P \quad (19c)$$

Once the constants c_j are obtained from the numerical solution of system (19), the solution vector is reconstructed from Eq. (19a). The inverse formula of the steady conjugated problem, Eq. (9b), is then recalled obtaining the dimensionless temperature at any (X, Y, Z) position, yielding the desired solution of the filter problem, and the steady state solution of the original problem itself.

Homogeneous transient conjugated problem

The homogeneous counterpart of the transient conjugated problem, after substituting the filtering proposal stated in Eq. (7) and the filter problem, Eq. (8), becomes the following partial differential equation:

$$W(X,Y) \left[\frac{\partial \theta_H}{\partial \tau} + U(X,Y) \frac{\partial \theta_H}{\partial Z} \right]$$

$$= \frac{4}{\sigma_x^2} \frac{\partial}{\partial X} \left[K(X,Y) \frac{\partial \theta_H}{\partial X} \right] + \frac{4}{\sigma_y^2} \frac{\partial}{\partial Y} \left[K(X,Y) \frac{\partial \theta_H}{\partial Y} \right]$$

$$+ \frac{1}{Pe^2} \frac{\partial}{\partial Z} \left[K(X,Y) \frac{\partial \theta_H}{\partial Z} \right],$$

$$0 < X < 2, 0 < Y < 2, 0 < Z < Z_\infty, \tau > 0 \quad (20a)$$

$$\theta_H(0,Y,Z,\tau) = 0, \theta_H(2,Y,Z,\tau) = 0, 0 \le Y \le 2, 0 \le Z \le Z_\infty, \tau > 0 \quad (20b,c)$$

$$\theta_H(X,0,Z,\tau) = 0, \theta_H(X,2,Z,\tau) = 0, 0 \le X \le 2, 0 \le Z \le Z_\infty, \tau > 0 \quad (20d,e)$$

$$\theta_H(X,Y,0,\tau) = 0, \left. \frac{\partial \theta_H}{\partial Z} \right|_{Z=Z_\infty} = 0, 0 \le X \le 2, 0 \le Y \le 2, \tau > 0 \quad (20f,g)$$

$$\theta_H(X,Y,Z,0) = 1 - \theta_P(X,Y,Z), 0 \le X \le 2, 0 \le Y \le 2, 0 \le Z \le Z_\infty \quad (20h)$$

In order to solve the homogeneous transient problem through integral transformation, the following transform-inverse pair is defined:

Transform:

$$\overline{\theta}_{H_i}(\tau) = \int_0^{Z_\infty} \int_0^2 \int_0^2 W(X,Y) \theta_H(X,Y,Z,\tau) \tilde{\Psi}_i(X,Y,Z) dX dY dZ \quad (21a)$$

Inverse:

$$\theta_H(X,Y,Z,\tau) = \sum_{i=1}^\infty \tilde{\Psi}_i(X,Y,Z) \overline{\theta}_{H_i}(\tau) \quad (21b)$$

The normalized eigenfunctions and the associated eigenvalues come from solving the following three-dimensional eigenvalue problem:

$$\frac{4}{\sigma_x^2} \frac{\partial}{\partial X} \left[K(X,Y) \frac{\partial \Psi_i}{\partial X} \right] + \frac{4}{\sigma_y^2} \frac{\partial}{\partial Y} \left[K(X,Y) \frac{\partial \Psi_i}{\partial Y} \right]$$

$$+ \frac{1}{Pe^2} \frac{\partial}{\partial Z} \left[K(X,Y) \frac{\partial \Psi_i}{\partial Z} \right]$$

$$+ \mu_i^2 W(X,Y) \Psi_i(X,Y,Z) = 0 \quad (22a)$$

$$\Psi_i(0,Y,Z) = 0, \Psi_i(2,Y,Z) = 0, 0 \le Y \le 2, 0 \le Z \le Z_\infty \quad (22b,c)$$

$$\Psi_i(X,0,Z) = 0, \Psi_i(X,2,Z) = 0, 0 \le X \le 2, 0 \le Z \le Z_\infty \quad (22d,e)$$

$$\Psi_i(X,Y,0) = 0, \left. \frac{\partial \Psi_i}{\partial Z} \right|_{Z=Z_\infty} = 0, 0 \le X \le 2, 0 \le Y \le 2 \quad (22f,g)$$

and the normalized eigenfunctions with the normalization integrals are calculated from:

$$\tilde{\Psi}_i(X,Y,Z) = \frac{\Psi_i(X,Y,Z)}{\sqrt{N_{\Psi_i}}}; N_{\Psi_i} = \int_0^{Z_\infty} \int_0^2 \int_0^2 W(X,Y) \Psi_i(X,Y,Z)^2 dX dY dZ \quad (22h,i)$$

As previously stated, the eigenvalue problem with space variable coefficients defined in Eq. (22) can be handled by the GITT itself with the proposition of a simpler auxiliary problem. Hence, the following auxiliary eigenvalue problem has been chosen:

$$\frac{4}{\sigma_x^2} \frac{\partial^2 \chi_k}{\partial X^2} + \frac{4}{\sigma_y^2} \frac{\partial^2 \chi_k}{\partial Y^2} + \frac{1}{Pe^2} \frac{\partial^2 \chi_k}{\partial Z^2} + \lambda_k^2 \chi_k = 0; 0 < X < 2, 0 < Y < 2, 0 < Z < Z_\infty \quad (23a)$$

$$\chi_k(0,Y,Z) = 0, \chi_k(2,Y,Z) = 0, 0 \le Y \le 2, 0 \le Z \le Z_\infty \quad (23b,c)$$

$$\chi_k(X,0,Z) = 0, \chi_k(X,2,Z) = 0, 0 \le X \le 2, 0 \le Z \le Z_\infty \quad (23d,e)$$

$$\chi_k(X,Y,0)=0, \frac{\partial \chi_k}{\partial Z}\bigg|_{Z=Z_\infty}=0, 0\leq X\leq 2, 0\leq Y\leq 2 \quad (23f,g)$$

As in the previous section, one can solve problem (23) via separation of variables, then employ a reordering scheme to rewrite the triple summation as a single one; each index then corresponds to the position of the triad of eigenvalues associated with each direction when sorted in ascending order of the sum of their squares. Thus, the following transform-inverse pair is proposed for the expansion of the original eigenfunctions:

Transform:

$$\overline{\Psi}_k^{(i)} = \int_0^{Z_\infty} \int_0^2 \int_0^2 \tilde{\chi}_k(X,Y,Z)\Psi_i(X,Y,Z)dXdYdZ \quad (24a)$$

Inverse:

$$\Psi_i(X,Y,Z) = \sum_{k=1}^\infty \tilde{\chi}_k(X,Y,Z)\overline{\Psi}_k^{(i)} \quad (24b)$$

where the normalized eigenfunctions and normalization integrals are given by the following expression:

$$\tilde{\chi}_k(X,Y,Z) = \frac{\chi_k(X,Y,Z)}{\sqrt{N_{\chi_k}}},$$
$$N_{\chi_k} = \int_0^{Z_\infty} \int_0^2 \int_0^2 \chi_k(X,Y,Z)^2 dXdYdZ \quad (24c,d)$$

The integral transformation of the differential eigenvalue problem with space variable coefficients, Eqs. (22a–g), is then achieved by operating with $\int_0^{Z_\infty} \int_0^2 \int_0^2 \tilde{\chi}_l(X,Y,Z) - dXdYdZ$, yielding the following algebraic eigensystem:

$$(\mathbf{D}+\mathbf{E})\overline{\Psi} = \mu^2 F\overline{\Psi} \quad (25a)$$

$$D_{kl} = -\int_0^{Z_\infty} \int_0^2 \int_0^2 [1-K(X,Y)]\left\{ \frac{4}{\sigma_x^2}\frac{\partial \tilde{\chi}_k}{\partial X}\frac{\partial \tilde{\chi}_l}{\partial X} + \frac{4}{\sigma_y^2}\frac{\partial \tilde{\chi}_k}{\partial Y}\frac{\partial \tilde{\chi}_l}{\partial Y} + \frac{1}{Pe^2}\frac{\partial \tilde{\chi}_k}{\partial Z}\frac{\partial \tilde{\chi}_l}{\partial Z} \right\}dXdYdZ \quad (25b)$$

$$E_{kl} = \lambda_k^2 \delta_{kl}; F_{kl} = \int_0^{Z_\infty} \int_0^2 \int_0^2 W(X,Y)\tilde{\chi}_k(X,Y,Z)\tilde{\chi}_l(X,Y,Z)dXdYdZ \quad (25c,d)$$

As previously discussed, well-established algorithms for matrix eigensystem analysis can be recalled to numerically solve the algebraic problem given by Eq. (25), yielding the eigenvalues μ and the eigenvectors $\overline{\Psi}$, after truncation to a large finite order N_H sufficient to warrant convergence to a desired accuracy target. Then, operating the homogeneous transient energy equation (20a) with $\int_0^{Z_\infty} \int_0^2 \int_0^2 \tilde{\Psi}_i(X,Y,Z) - dXdYdZ$ and making use of the original eigenvalue

problem equation, Eqs. (22a–g), and the inverse formula (21b), one obtains the following initial value problem of ordinary differential equations for the homogeneous transient transformed temperatures:

$$\frac{d\overline{\theta}_{H_i}}{d\tau} + \mu_i^2 \overline{\theta}_{H_i}(\tau) + \sum_{j=1}^\infty \overline{G}_{i,j}\overline{\theta}_{H_j}(\tau) = 0, i = 1,2,3,\ldots \quad (26a)$$

$$\overline{G}_{i,j} = \int_0^{Z_\infty} \int_0^2 \int_0^2 W(X,Y)U(X,Y)\tilde{\Psi}_i(X,Y,Z)\frac{\partial \tilde{\Psi}_j}{\partial Z}dXdYdZ \quad (26b)$$

with the transformed initial conditions:

$$\overline{\theta}_{H_i}(0) = \overline{f}_i = \int_0^{Z_\infty} \int_0^2 \int_0^2 W(X,Y)[1-\theta_P(X,Y,Z)]$$
$$\tilde{\Psi}_i(X,Y,Z)dXdYdZ \quad (26c)$$

The infinite system is truncated to a sufficiently large order N_T for the desired precision target and, again, the transformed system of equations (26a–c), despite being coupled, is linear and has analytical solution through the appropriate matrix eigensystem analysis. For this purpose, it is written in matrix form as:

$$\overline{\theta}'_H(\tau) + \mathbf{G}\overline{\theta}_H(\tau) = 0; \overline{\theta}_H(0) = \overline{\mathbf{f}} \quad (27a,b)$$

where $\mathbf{G}$, the coefficients matrix independent from the time variable, and the transformed initial conditions vector, $\overline{\mathbf{f}}$, are given by:

$$\mathbf{G} = \left\{ \mu_i^2 \delta_{i,j} + \overline{G}_{i,j} \right\}_{i,j=1,2,3,\ldots}; \overline{\mathbf{f}} = \left\{ \overline{f}_i \right\}_{i=1,2,3,\ldots} \quad (27c,d)$$

The solution for the transformed temperatures vector is then conveniently written in terms of the matrix exponential function as:

$$\overline{\theta}_H(\tau) = \exp(-\mathbf{G}\tau)\cdot f \quad (28)$$

The inversion formula of the homogeneous transient conjugated problem, Eq. (21b), is then recalled, together with the filter proposal, Eq. (7), to provide the dimensionless temperature field for any dimensionless position and time, (X,Y,Z,τ), leading to the desired solution of the transient three-dimensional conjugated problem. Once the temperature distribution has been derived, the transient bulk temperature along the channel length is given from its definition as:

$$\theta_{av}(Z,\tau) = \frac{\int_A \theta(X,Y,Z,\tau)U(X,Y)dA}{\int_A U(X,Y)dA} \quad (29)$$

Table 1. Thermophysical properties of the working fluid and substrates.

Thermophysical Properties at 20 °C			
	ρ (kg/m³)	c_p (kJ/kg K)	k (W/m K)
Water	998	3.18	0.60
PDMS	970	1.46	0.15
Fused Silica	2220	0.74	1.38
Commercial Copper (90% Cu, 10% Al)	8800	0.42	52

Table 2. Geometric parameters for the test case of a square channel ($AR = 1$).

Parameter	Value	Parameter	Value
L_{x_M}	100 μm	D_h	100 μm
L_{y_M}	100 μm	σ_x	2
L_{x_T}	200 μm	σ_y	2
L_{y_T}	200 μm	Pe	1

Table 3. Convergence of the first ten eigenvalues of the two-dimensional eigenvalue problem in the case of a square channel ($AR = 1$).

B	N_F=200	N_F=400	N_F=600	N_F=800	N_F=1000
1	1.4208	1.4162	1.4147	1.4136	1.4128
2	3.3352	3.3345	3.3341	3.3339	3.3338
3	3.3352	3.3345	3.3341	3.3339	3.3338
4	4.4429	4.4429	4.4429	4.4429	4.4429
5	5.3523	5.3380	5.3336	5.3303	5.3278
6	5.5828	5.5720	5.5686	5.5662	5.5643
7	5.9380	5.9263	5.9206	5.9176	5.9153
8	5.9380	5.9263	5.9206	5.9176	5.9153
9	6.6181	6.5938	6.5772	6.5705	6.5653
10	6.6181	6.5938	6.5772	6.5705	6.5653

Results and discussion

Convergence analysis and verification

The fully developed velocity profile is taken from well-known exact analytical solutions [36], as presented in Eqs. (6a–f). For this reason, the focus of the work is directed toward analyzing the temperature field and other derived quantities, in order to probe the adequacy of the integral transforms approach here proposed.

In order to identify typical values of the dimensionless governing parameters, one fluid and three substrate materials were selected from previous experiments, as shown in Table 1, with water as working fluid and PDMS (polydimethylsiloxane), Fused Silica and Commercial Copper (90% Cu, 10% Al) as substrates, first of all seeking comparisons with the analysis presented in Ref. [55]. The chosen geometric parameters for a square channel, which are employed to characterize the dimensionless parameters, are presented in Table 2. The results that follow encompass temperature profiles in the transient and steady regimen for the cases of the squared cross section ($AR = 1$) and rectangular channels with different wall thicknesses, for fixed channel width and variable channel height.

Table 4. Convergence of the first ten eigenvalues of the three-dimensional eigenvalue problem in the case of a square channel ($AR = 1$).

μ	N_H=200	N_H=400	N_H=600	N_H=800	N_H=1000
1	1.4482	1.4330	1.4323	1.4317	1.4314
2	1.6878	1.6746	1.6740	1.6735	1.6732
3	2.0856	2.0747	2.0743	2.0739	2.0737
4	2.5681	2.5597	2.5594	2.5591	2.5590
5	3.0957	3.0948	3.0891	3.0889	3.0889
6	3.1948	3.1939	3.1919	3.1919	3.1918
7	3.1948	3.1939	3.1919	3.1919	3.1918
8	3.3044	3.3033	3.3014	3.3014	3.3014
9	3.3044	3.3033	3.3014	3.3014	3.3014
10	3.5130	3.5117	3.5100	3.5100	3.5099

Table 5. Convergence of the dimensionless temperature profile for the case of the squared channel ($AR = 1$), at the centerline ($X = 1$ and $Y = 1$) of the microfluidic apparatus.

$\tau = 0.5$							
N_T	$Z = 0.1$	$Z = 0.2$	$Z = 0.3$	$Z = 0.5$	$Z = 0.75$	$Z = 1$	$Z = 2$
30	0.9591	0.9182	0.8709	0.7766	0.6764	0.6031	0.4908
60	0.9585	0.9171	0.8695	0.7754	0.6761	0.6032	0.4914
90	0.9578	0.9157	0.8675	0.7725	0.6722	0.5980	0.4820
120	0.9575	0.9153	0.8671	0.7725	0.6725	0.5981	0.4820
150	0.9567	0.9137	0.8648	0.7690	0.6672	0.5905	0.4665
180	0.9567	0.9136	0.8649	0.7693	0.6675	0.5905	0.4665
COMSOL	0.9596	0.9144	0.8668	0.7739	0.6745	0.6006	0.4889
Rel. Deviation (%)	0.31	0.08	0.18	0.47	0.9	1.5	4.1

In order to analyze the convergence of the eigenfunction expansions for the temperature distribution, one needs first to evaluate the convergence of the two and three-dimensional eigenvalue problems solutions, based on the integral transformation with simpler auxiliary eigenvalue problems. Thus, Tables 3 and 4 illustrate the convergence of the 2D and 3D eigenvalue problems in the case of a square channel. The presented first ten eigenvalues converge to at least the third significant digit with truncation orders as low as N_F or N_H=1000, where it is noticed a slower convergence rate, as expected, relative to the two-dimensional eigenvalue problem in Ref. [55], given that the symmetry of the problem has been accounted for beforehand in that work, thus reducing the number of terms required for convergence. The convergence behavior of the eigenvalue problems expansions can be enhanced by employing, for instance, an extended integral balance approach for Sturm-Liouville problems [67].

Table 5 offers a brief convergence analysis of the transient dimensionless temperature at the centerline ($X = 1$ and $Y = 1$) and time $\tau = 0.5$, for selected values of the axial coordinate. The centerline dimensionless temperature is converged to ±1 in the fourth significant digit, for truncation orders up to N_T=180 terms. Also shown are the results from the commercial software
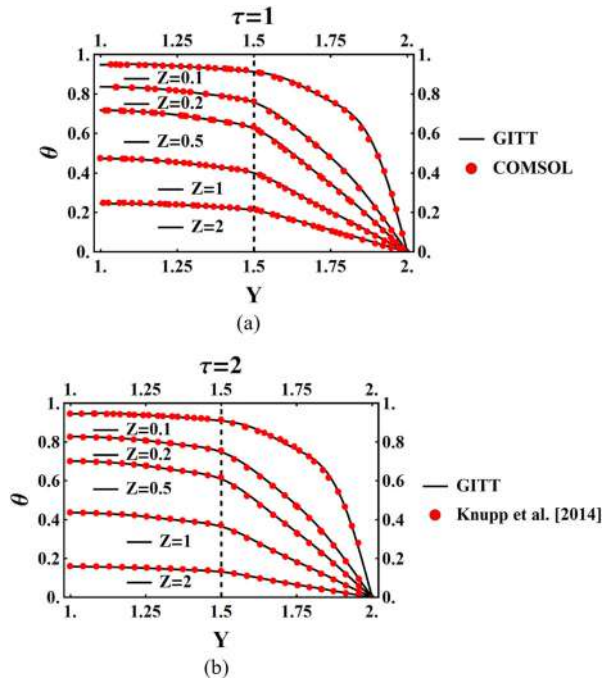
Figure 2. Dimensionless temperature profiles in transient state for a squared channel: (a) at $\tau = 1$, compared with COMSOL solution and (b) At steady state ($\tau = 2$) compared with the GITT solution with partial transformation scheme [55], across the transversal coordinate, $Y \in [1, 2]$, at the central plane ($X = 1$) and for different axial positions.

Figure 3. Dimensionless temperature profiles in transient state for a water/PDMS squared channel at its centerline ($Y = 1$ and $X = 1$), compared with COMSOL solution for different axial positions.

Figure 4. Dimensionless temperature profiles at steady state for a rectangular channel with aspect ratio $AR = 1.42$, compared against the GITT solution with partial transformation of Knupp et al. [55]: (a) along the width of the microfluidic apparatus, at the center plane ($Y = 1$) and (b) along the height of the micro-device, at the center plane ($X = 1$), for different axial positions.

COMSOL Multiphysics® finite element method simulation, with corresponding relative deviations.

Figure 2a,b shows the dimensionless temperature profiles in a square channel with water as working fluid and PDMS as substrate, across the transversal coordinate, $Y = 1$ to 2, at the central plane $X = 1$, and for different times ($\tau = 1$ and 2), where the steady state is approximately represented by the larger time value ($\tau = 2$). Different axial positions along the microchannel length are plotted, where it is possible to clearly distinguish the transition at the fluid/solid interface, identified by a dashed line to highlight the position of the interface. The GITT solution at $\tau = 1$ is verified against the solution of the commercial software COMSOL Multiphysics®, whereas the solution at $\tau = 2$ is verified against the steady state GITT solution with partial transformation of Knupp et al. [55], both represented as red circles, showing excellent agreement to the graph scale in both cases.

Figure 3 shows the dimensionless temperature time evolution at the centerline of the microfluidic apparatus ($Y = 1$ and $X = 1$), located within the fluid domain, and compared with the COMSOL Multiphysics® results. Clearly, the GITT not only predicts well the spatial behavior of the micro-device, but also its temporal behavior, with only $N_T = 180$ terms, which is a

fairly low truncation order for the eigenfunction expansion, given the complexity of the problem (three-dimensional transient conjugated formulation).

Figure 4 shows the steady state ($\tau = 2$) dimensionless temperature profiles for a rectangular channel with aspect ratio $AR = 1.42$, again across the transversal coordinate, $Y = 1$ to 2, at the central plane $X = 1$ of the channel-substrate arrangement, for different longitudinal positions. Following Knupp et al. [55], the width of the channel is fixed while the height is varied with respect to the square channel case. The

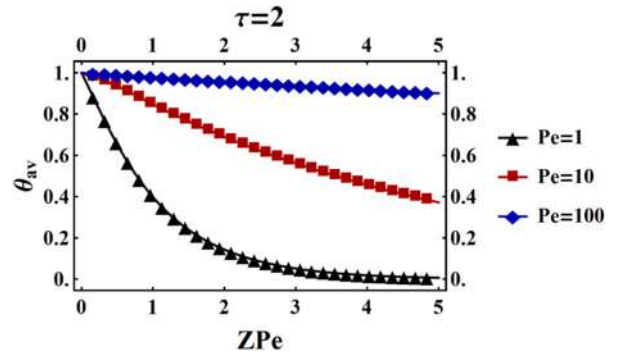**Table 6.** Materials and conductivities ration considered in the present work.

Materials	Thermal conductivities ratio k_s/k_f
PDMS/Water	0.25
Silica/Water	2.3
Copper/Water	86.66

Table 7. Maximum dimensionless temperature variation across the channel.

$\Delta\theta_{cw}$			
Materials	$Z = 0.5$	$Z = 1$	$Z = 1.5$
PDMS/Water	0.110	0.086	0.053
Silica/Water	0.066	0.031	0.010
Copper/Water	0.025	0.008	0.001

Figure 5. Dimensionless bulk temperature along the scaled dimensionless axial coordinate for different Peclet numbers, with $AR = 1$ and $\tau = 2$.

present GITT solution obtained with the total transformation scheme has an excellent agreement with the GITT solution obtained by Knupp et al. [55] with the pseudo-transient formulation and the partial transformation scheme, when a two-dimensional eigenvalue problem was adopted and a partial differential system for the transformed temperatures, with time and longitudinal coordinate as independent variables, is numerically solved for. The proposed total transformation scheme, with its transformed problem consisting of a system of ordinary differential equations solvable by the associated matrix eigensystem analysis, bears a noticeable advantage when compared with the transformed partial differential equations system stemming from the partial transformation alternative strategy adopted in Ref. [55], both in terms of computational cost and numerical accuracy. A more complete comparative analysis of the total and partial transformation schemes in multidimensional convection-diffusion problems was provided in Ref. [68]. The adherence of these results confirms not only the robustness of the integral transform approach, but also its flexibility in terms of solution path choices.

Table 6 provides the values of the thermal conductivity ratio, an important parameter in conjugated heat transfer, for the three different substrates cases here considered. Table 7 shows the maximum dimensionless temperature variation across the wall, $\Delta\theta_{cw}$, for the three cases and at three different longitudinal positions, which shows that, as expected, there is a more marked temperature variation across the height of the channel closer to the channel entry ($Z = 0.5$) and for smaller thermal conductivity ratio. The higher thermal conductivities ratio, for fixed Peclet number and aspect ratio, increases the conjugation parameter as defined in Refs. [29, 46] and favors the adoption of the approximate lumped-differential model that lumps the temperature distribution in the transversal direction at the wall, due to the milder transversal temperature gradients, and accounts for axial conduction only in the solid region.

Figure 5 illustrates the effect of Peclet number on the dimensionless bulk temperature distribution at the fluid region of the microfluidic apparatus, across the scaled axial coordinate ZPe, for $AR = 1$, again with the fluid/solid combination of water/PDMS. For lower Pe numbers, the increased relevance of fluid axial conduction alters the usual proportionality between thermal entry length and hydraulic diameter ratio with the Peclet number. Nevertheless, the conjugation effect does not alter the same qualitative behavior, i.e., larger thermal entry lengths with rising Pe, as can be extracted from the comparison of the curves for $Pe = 1$ to 100 in Figure 5, which stems from the lower residence times of the fluid within the channel offsetting the wall participation in heat transfer.

Figure 6 shows the dimensionless fluid bulk temperature at steady state along the axial coordinate, now varying the aspect ratio AR for a fixed $Pe = 1$, again utilizing water as working fluid and PDMS as substrate. It can be observed the more effective cooling with the reduction of the aspect ratio, while maintaining the total fluid-solid interface area.

Figure 7 depicts dimensionless temperature contours at the midplane parallel to YZ for four different values of aspect ratio, defined as the ratio between the height and the width of the channel. For an increase in aspect ratio, the wall thickness becomes smaller which is observable in Figure 7 by recognizing the purple horizontal line as the interface between the fluid and the solid. Close to the channel entrance section, the temperature assumes its highest values, as imposed by the boundary condition at $Z = 0$, both at the fluid and the wall; as the flow advances along the channel, the temperature rapidly decays as the flow
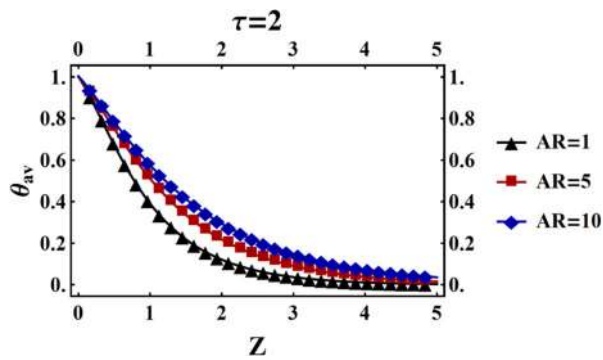
Figure 6. Dimensionless fluid bulk temperature along the axial coordinate for different aspect ratio, *AR*, for the same total area of the fluid-solid interface.

approaches full thermal development. Along the *Y* axis, the temperature decreases monotonically from the center of the channel to the edge of the solid wall. One may observe the transitions from the fluid to the solid regions, in accordance with the interface continuity conditions. The single domain formulation allows for the representation of the full region temperature distribution through a single eigenfunction expansion, which is indeed a major advantage of this solution path and permits envisioning the extension of this approach to more involved geometries and complex configurations. Besides, this precise and cost-effective simulation path is particularly handy in
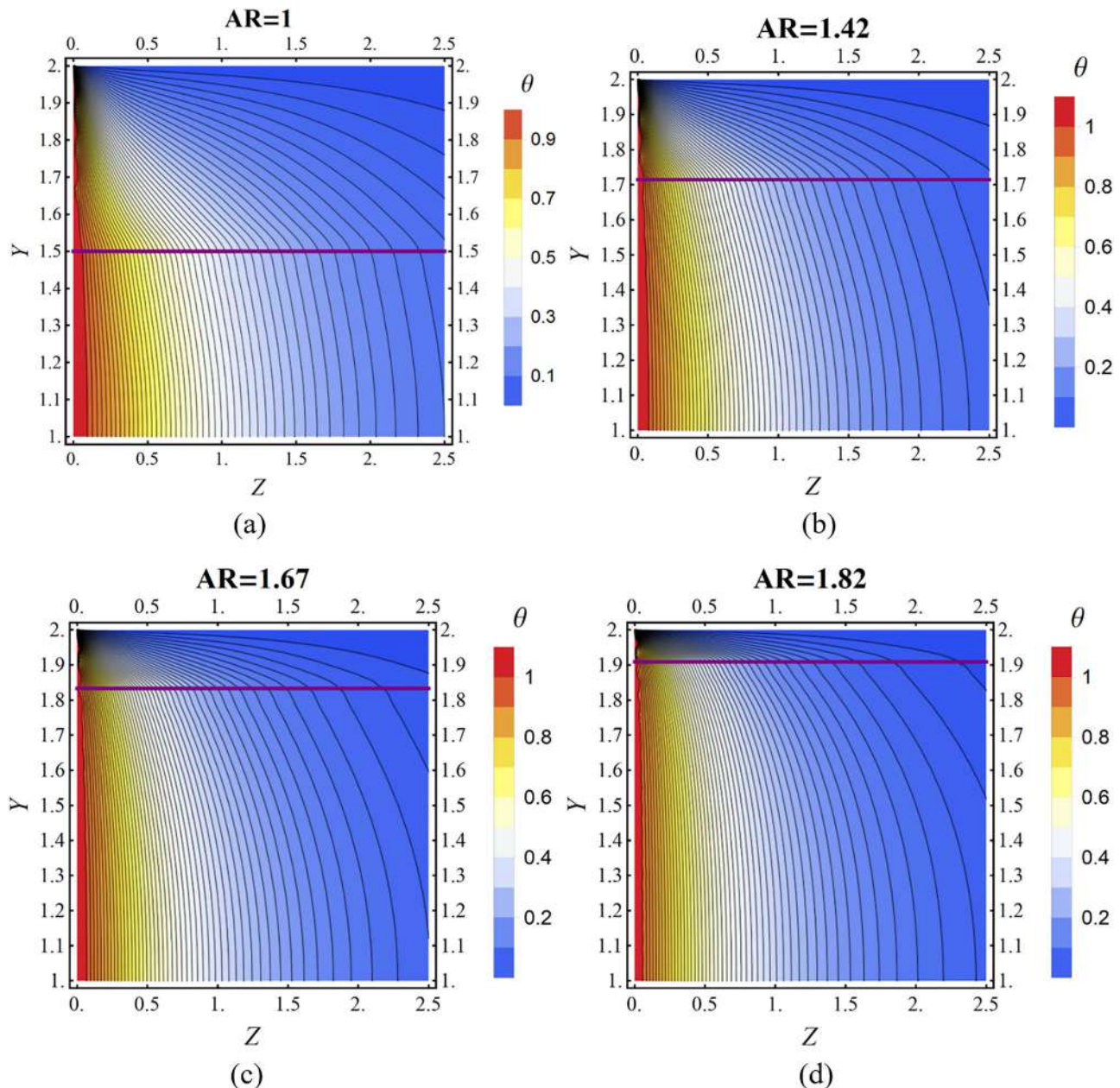
Figure 7. Dimensionless temperature contours along the axial coordinate and across the width of the microsystem, at the center plane (*X* = 1): (a) *AR* = 1, (b) *AR* = 1.42, (c) *AR* = 1.67 and (d) *AR* = 1.82.

intensive computational tasks such as inverse problem analysis of heterogeneous media, especially under the integral transformation of experimental data, as demonstrated in Refs. [69, 70].

Conclusions

The present work has analyzed transient three-dimensional conjugated heat transfer for internal laminar flow in fluidic devices based on microchannels. The proposed hybrid solution was constructed by the GITT, reformulating the problem in a single domain for the fluid current and solid substrate present on the original formulation, which are represented through space variable thermophysical properties and velocity fields, and considering both transversal and axial diffusion effects at both the wall and fluid subdomains. The spatially variable coefficients of the single domain formulation are carried on to a two-dimensional eigenvalue problem for the steady state solution, and to a three-dimensional eigenvalue problem for the transient solution, which give the basis of the corresponding eigenfunction expansions and are also solved via GITT. The results for a single rectangular microchannel show excellent agreement both on the transient regime against the results obtained by the commercial platform COMSOL, and on the steady regime with the solution obtained by Knupp et al. [55], also handled by GITT, in which the authors utilized a partial transformation scheme and a pseudo-transient formulation. The present analysis also inspected for the influence on the conjugated heat transfer behavior of parameters such as thermal conductivity ratio, aspect ratio, and Peclet number. The present contribution closes a gap on the integral transform treatment of conjugated heat transfer problems, by dealing with a fairly general transient three-dimensional situation, but also opens avenues for extensions in terms of channel transversal and longitudinal geometries, micro-device multichannel configuration, flow and thermal regimen, and micro-heat transfer scaling effects.

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