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Unified Integral Transforms and Non-Classical Eigenvalue Problems in Heat and Mass Transfer

The generalized integral transform technique (GITT) is reviewed as a computational–analytical methodology in linear and nonlinear convection–diffusion problems, based on eigenfunction expansions extracted from characteristic differential operators, coefficients, and boundary conditions present in the original partial differential problem formulation. Here, the emphasis is on the employment of nonclassical eigenvalue problems as the expansion basis, which do not fall into the more usual framework of Sturm–Liouville problems. The goal is to enable or improve the eigenfunction expansions convergence, by incorporating more information from the original operators into the chosen eigenvalue problem, while requiring the handling of such a more involved expansion base. In this concern, the proposed differential eigenvalue problem can itself be handled by the GITT, leading to an algebraic eigensystem analysis. Different classes of nonclassical eigenvalue problems are then reviewed and associated with typical applications in heat and mass transfer. Representative test cases are then chosen to illustrate the extended methodology and demonstrate the convergence rates attainable by this enhanced hybrid solution path. [DOI: 10.1115/1.4055818]

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1 Introduction

Modern engineering practice is nowadays inconceivable without the computational simulation tools that were introduced and advanced over the last few decades, both in individual physical areas and in the Multiphysics framework. Models, methodologies, and algorithms were combined either as a standalone tool or as parts of more general multipurpose packages. While much has been achieved and progressively consolidated, offering to the specialized practitioner an arsenal of solution alternatives, the research on numerical methodologies and analysis in engineering sciences simulation is far from being exhausted.

Hybrid numerical–analytical methods for partial differential equations, in different physical contexts, are recognized not only as a benchmarking tool for numerical schemes and codes but also as alternative computational–analytical methodologies themselves, that bridge the gap between simple mathematical formulations, that are directly treatable by classical analytical methods, and more complex formulations, that in general require costly numerical methods and associated computational codes [1–3]. Such hybrid approaches are derived from classical analytical methods, mostly developed before the computer boom in science and engineering. In the transport phenomena area, the integral transform method is certainly the most widely applied analytical approach, and a few reference books have been made available over the years [4–9]. One such extension of classical methods is the so-called generalized integral transform technique (GITT) [10–17], which has been previously reviewed with different application emphasis [1–3,18–20].

The GITT is an eigenfunction expansion approach for partial differential equations, which relies on the choice of an eigenvalue problem that desirably carries enough characteristic information on the original problem differential operators and coefficients, to

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warrant convergence of the expanded potential, within practical limits, of the infinite eigenfunction expansion truncation order. Eventually, convergence acceleration techniques, such as analytical filtering and/or integral balance schemes, might be recalled, offering improved convergence and, consequently, reduced computational costs. In the realm of uncoupled diffusion problems with the most common first to third-type boundary conditions, Sturm–Liouville's theory provides a natural choice of eigenfunction expansion base, and it has been the preferred eigenvalue problem proposition in various applications reported in the literature. However, when equation and/or boundary conditions coupling among the different potentials become a dominant effect, the simpler choice of decoupled Sturm–Liouville-type problems might not be the best possible path for constructing the analytical representation. The treatise by Mikhailov and Özisik [9] unifies and classifies linear diffusion problems in seven different classes of formulations, which can be solved exactly through integral transforms, starting with the so-called Class I problems that involve essentially diffusion of a single potential. Then, in their systematic presentation, the following classes of problems evolve in terms of linear coupling terms among the different potentials, which may be present at either or both the governing equations and boundary conditions. Exact solutions are then provided by integral transforms, starting from the nonclassical eigenvalue problems that are obtained through the separation of variables as applied to the homogeneous versions of the originally posed problems. This compendium [9] is a unique source for both the integral transform approach and ready-to-use exact expressions for a wide range of problems. Meanwhile, other classes of linear problems were also explored, and exact solutions were achieved through integral transforms based on nonclassical eigenvalue problems, though not compiled in one single source, such as in the case of boundary conditions with a finite capacitance [21], diffusion in anisotropic media [22,23], and extended Gräetz problems [24,25], to name a few. Also, the advancement of the GITT promoted different alternatives of eigenfunction expansion basis, such as in handling eigenvalue problems in irregular domains [26,27], solving moving boundary problems [28,29], in extensions to fourth-order eigenvalue problems for boundary layer and Navier–Stokes equations [30,31], in single domain representation of heterogeneous multiregion problems [32,33], in incorporating convective terms into the eigenvalue problems [34,35], in vector eigenfunction expansions for Navier–Stokes equations [36,37], and more recently, in accounting for nonlinearities in the eigenvalue problem itself [38,39].

The present contribution first provides an overview of the GITT as a computational–analytical tool in linear and nonlinear convection–diffusion problems, including the solution of the required eigenvalue problem itself. Emphasis is then placed on reviewing different classes of eigenfunction expansion proposals that extend the classical Sturm–Liouville theory, pointing out the novel aspects in each alternative solution path. The aim is to provide a unified compilation source for developments that have either enabled or offered relevant computational gains in convergence, as opposed to the conventional approach of choosing a classical Sturm–Liouville auxiliary problem. Finally, selected applications are more closely examined, to illustrate such convergence rates and allow for discussions of their relative merits, while accounting for the increased analytical complexity.

2 Basic Steps and Formal Solutions

Before describing the formalism behind the GITT approach [1–3] for a certain proposed partial differential problem formulation, it is worthwhile to review the basic steps that should be followed in its application to a new problem, and thus offer an overview of the methodology for new users. The GITT is an eigenfunction expansion approach that is not restricted to formulations that permit finding an exact decoupled analytical solution, such as in the classical integral transform method [9] for linear transformable diffusion problems. Thus, it offers some alternative solution paths that the user can benefit from. As general guidance, the basic steps can then be summarized as:

(1) Expansion base selection: This very first step is perhaps the one that requires the most experience with the hybrid methodology since it somehow defines the effort balancing between analytical and numerical tasks. Essentially, if more information from the original problem operators is carried into the eigenvalue problem that provides the expansion base, faster expansion convergence, and more weakly coupled transformed systems are expected. Thus, in general, that means more analytical and less numerical work. Even though heterogeneous and anisotropic media, moving boundaries, and nonlinear eigenvalue problems can nowadays be considered, the user should judge its capability or need of handling a more involved eigenvalue problem, instead of opting for a simpler expansion base and then just handling larger transformed ordinary differential equations (ODE) systems for the required convergence criteria.

(2) Problem rewriting: Once the eigenvalue problem has been selected, it is handy to rewrite the original problem formulation by showing explicitly the chosen operators that are part of the eigenvalue problem formulation, since these will be exactly transformed in the step ahead, while all the remaining terms are gathered into an expanded source term that will be transformed in a single integral operation. For instance, if for a certain nonlinear problem, it has been chosen to employ a linear eigenvalue problem, all the corresponding nonlinear portions of the terms in the original formulation are then moved to the corresponding source terms in the equations and boundary conditions.

(3) Filtering: An important step for reducing the computational effort, analytical filtering can extract information from the original formulation that corresponds to a portion of the solution or for an approximate formulation of the original problem that allows for an analytical solution. In any integral transform solution, the source terms are responsible for deviating the convergence rate pattern from the pure exponential spectral-type behavior. The more important the source terms, the more impacted shall be the convergence rates; thus, filtering can eventually eliminate source terms or at least reduce their importance in the filtered problem formulation. It should be recalled that the concept of the source term in the present context goes beyond the physical meaning of a source/sink effect, also incorporating all the mathematical terms that are not exactly transformed through the specific choice of eigenvalue problem, and then have to be merged into the expanded source terms. It is already quite useful to filter at least the boundary conditions, providing a more uniform convergence behavior throughout the solution domain. In addition, successive filtering remains a useful tool in improving convergence rates once a single filter pass is not judged to enhance convergence to the desired rates.

(4) Eigenvalue problem solution: Classical separation of variables is indeed an important tool in handling simpler eigenvalue problems, especially with the aid of modern symbolic computation platforms. However, as more involved formulations are considered by retaining more information from the original partial differential equation (PDE), as advocated above, it is unlikely that an exact analytical solution can be readily obtained. Then, the GITT itself is a versatile approach in handling such eigenvalue problems, expanding the solution in terms of simpler eigenfunctions, thus leading to algebraic eigenvalue problems in the transformed domain that can be handled numerically by well-established algorithms and routines.

(5) Integral transformation process: This step is essentially the application of the appropriate integral transformation

operator over the resulting filtered problem formulation, manipulated together with the boundary conditions once source terms are present. From the substitution of the inverse formula within such integrals, the outcome is the need for evaluating the integral coefficients, preferably in analytical form, again with the aid of symbolic computation systems. Once the integrand does not allow for a fully analytical integration, such as for most nonlinear formulations, the alternative of a semi-analytical integration scheme is the preferable path, thus avoiding costly numerical integration routines, especially when called from within the transformed ODEs system solver.

(6) Transformed system solution: Linear PDEs with space variable-only coefficients, in general, allow for an analytical solution of the transformed potentials system, even if it is a coupled infinite system, through the appropriate truncation and handling via the corresponding matrix eigensystem analysis. Otherwise, the numerical solution of the transformed ODEs system is in general the most time-consuming task in this hybrid numerical–analytical methodology. The transformed system is likely to be stiff in the numerical analysis sense, but several routines are readily available, either in the public domain or general-purpose libraries that can be successfully employed under reliable accuracy control schemes.

(7) Recovery of original potentials: Once the transformed potentials are either analytically or numerically obtained, the inverse formula of the integral transform pair is recalled to reconstruct the filtered potential in analytic form, and then added to the filter solution to reconstruct the desired original potentials.

The emphasis in the present review is on the employment of nonclassical eigenvalue problems as the expansion base, either due to the impossibility of adopting a classical Sturm–Liouville problem in a particular class of problems or to the actual choice of incorporating more information into the eigenvalue problem, from the original formulation operators, for improved computational performance. For more clarity in the classes of problems that may be dealt with through this alternative procedure, first, the most usual solution path of the GITT, adopting classical Sturm–Liouville problems as a base for the eigenfunction expansions, is briefly reviewed. We consider a sufficiently general nonlinear convection–diffusion problem for the n potentials, $T_k(\mathbf{x},t)$, $k=1,2,\ldots n$, which encompasses most classes of linear problems considered in [9] and some of the additional extended formulations here discussed, as special cases, defined in the region V with boundary surface S as

$$w_k(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial t}=\nabla\cdot\left[K_k(\mathbf{x})\nabla T_k(\mathbf{x},t)\right]-d_k(\mathbf{x})T_k(\mathbf{x},t)+P_k(\mathbf{x},t,\mathbf{T}),\quad \mathbf{x}\in V,\ t>0 \tag{1a}$$

with initial and boundary conditions given by

$$T_k(\mathbf{x},0)=f_k(\mathbf{x}),\quad \mathbf{x}\in V \tag{1b}$$

$$\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]T_k(\mathbf{x},t)=\phi_k(\mathbf{x},t,\mathbf{T}),\quad \mathbf{x}\in S,\ t>0 \tag{1c}$$

$$\mathbf{T}=\{T_1,T_2,\ldots,T_k,\ldots,T_n\}^T \tag{1d}$$

It should be clarified that Eq. (1) is not necessarily the original form of the proposed problem, but already encompass the basic steps (1) and (2) above, reflecting the user choice of eigenvalue problem through the coefficients in the operators of Eq. (1), which in the present representation are linear and dependent only on the spatial coordinates. Therefore, through the coefficients, $w_k(\mathbf{x})$, $K_k(\mathbf{x})$, $d_k(\mathbf{x})$, $\alpha_k(\mathbf{x})$, $\beta_k(\mathbf{x})$, a choice of the characteristic functional behavior to be represented through the eigenfunction expansion base has been implicitly made in rewriting the problem as in Eq. (1). Thus, a nonlinear eigenvalue problem is not explicitly considered in this formal analysis, but the present choice still allows for dealing with nonlinear formulations through the source terms in both the governing equations and boundary conditions, which incorporate the nonlinear terms of the original formulation that have been merged into the expanded source functions, $P_k(\mathbf{x},t,\mathbf{T})$ and $\phi_k(\mathbf{x},t,\mathbf{T})$, as described in more detail in Ref. [3]. For linear source terms, $P_k(\mathbf{x},t)$ and $\phi_k(\mathbf{x},t)$, Eq. (1) becomes the Class I problem according to the classification in Ref. [9], for which exact analytical solutions are readily available through the integral transform method. It is also implicit that Eq. (1) represents either the unfiltered or filtered formulations, since the filtering process essentially modifies the functional form of the equation and boundary source terms, $P_k(\mathbf{x},t,\mathbf{T})$ and $\phi_k(\mathbf{x},t,\mathbf{T})$, and the initial condition, $f_k(\mathbf{x})$.

Following the basic steps in the formal solution, the problem formulation in Eq. (1) already suggests the decoupled Sturm–Liouville eigenvalue problems to be considered, given by

$$\nabla\cdot[K_k(\mathbf{x})\nabla\psi_{ki}(\mathbf{x})]+[\mu_{ki}^2 w_k(\mathbf{x})-d_k(\mathbf{x})]\psi_{ki}(\mathbf{x})=0,\quad \mathbf{x}\in V,k=1,2,\ldots n \tag{2a}$$

$$\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]\psi_{ki}(\mathbf{x})=0,\quad \mathbf{x}\in S \tag{2b}$$

where the eigenvalues, μ_{ki}, and eigenfunctions, $\psi_{ki}(\mathbf{x})$, are known from the application of separation of variables or from the GITT itself [14,18]. The orthogonality property of the general Sturm–Liouville eigenfunctions provides the following integral transform pairs:

$$\bar{T}_{ki}(t)=\int_V w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x})T_k(\mathbf{x},t)\mathrm{d}V,\quad \text{transforms} \tag{3a}$$

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}\tilde{\psi}_{ki}(\mathbf{x})\bar{T}_{k,i}(t),\quad \text{inverses} \tag{3b}$$

where the normalized eigenfunctions and respective norms are given by

$$\tilde{\psi}_{ki}(\mathbf{x})=\frac{1}{\sqrt{N_{ki}}}\psi_{ki}(\mathbf{x}),\quad \text{with} \tag{3c}$$

$$N_{ki}=\int_{\mathrm{v}} w_k(\mathbf{x})\psi_{ki}^2(\mathbf{x})\mathrm{d}V \tag{3d}$$

By applying the operator $\int_V\tilde{\psi}_{ki}(\mathbf{x})(.)\mathrm{d}V$ over Eq. (1a) and making use of the boundary conditions, Eqs. (1c) and (2b), the coupled transformed system is obtained as

$$\frac{d\bar{T}_{ki}(t)}{dt}+\mu_{ki}^2\bar{T}_{ki}(t)=\bar{g}_{ki}(t,\bar{\mathbf{T}}),\quad i=1,2,\ldots,\ t>0,\quad k=1,2,\ldots,n \tag{4a}$$

while the initial conditions, Eq. (1b), are transformed through the operator $\int_V w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x})(.)dV$

$$\bar{T}_{ki}(0)=\bar{f}_{ki}\equiv\int_V w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x})f_k(\mathbf{x})dV \tag{4b}$$

and the transformed source terms, $\bar{g}_{ki}(t,\bar{\mathbf{T}})$, become

$$\bar{g}_{ki}(t,\bar{\mathbf{T}})=\int_V\tilde{\psi}_{ki}(\mathbf{x})P_k(\mathbf{x},t,\mathbf{T})dV+\int_S\phi_k(\mathbf{x},t,\mathbf{T})\left[\frac{\tilde{\psi}_{ki}(\mathbf{x})-K_k(\mathbf{x})\frac{\partial\tilde{\psi}_{ki}(\mathbf{x})}{\partial\mathbf{n}}}{\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})}\right]dS \tag{4c}$$

The formal solution of the transformed potentials is given by

$$\bar{T}_{ki}\left(t\right) = \bar{f}_{ki}\exp(-\mu_{ki}^2 t) + \int_0^t \bar{g}_{ki}\left(t', \bar{\mathbf{T}}\left(t'\right)\right)\exp\left[-\mu_{ki}^2(t-t')\right]dt' \tag{5}$$

Again, for linear source terms, Eq. (5) becomes the exact solution for the transformed potentials, recovering the general solution for Class I problems in Ref. [9]. For nonlinear problems, Eq. (5) is still useful in the inspection of convergence rates or providing approximate estimates, through linearization of the transformed source terms. In the more general nonlinear situations, the resulting coupled system (4) can be numerically solved, after being truncated to a sufficiently large finite order, N_k, for each potential. Well-tested initial value problem solvers that include the option of handling stiff systems, preferably with automatic accuracy control, should then be employed. The *Mathematica* system [40] provides the NDSolve function, which automatically implements an interpolating function object that recovers the behavior of the transformed potential along with the t variable as a continuous function. Then, the inverse formula (3*b*) provides the analytical representation of the desired potentials.

The hybrid numerical–analytical solution is only completed once the eigenvalue problem 2 has been solved for the eigenvalues and respective eigenfunctions. Symbolic computation platforms [40] have been facilitating the job of finding analytical solutions for separable Sturm–Liouville problems, but this possibility shall not be further discussed here. The most general case of an eigenvalue problem with an unknown analytical solution shall be addressed instead. The same hybrid approach, GITT, can be used to handle this general differential eigenvalue problem. Through the choice of a simpler auxiliary eigenvalue problem with a known exact solution, the eigenfunction expansion approach reduces problem (2) to an algebraic eigensystem analysis [14,18,27,41,42]. This auxiliary eigenvalue problem, for each potential order k, is given as

$$\nabla\cdot[\hat{K}_k(\mathbf{x})\nabla\Omega_{km}(\mathbf{x})] + [\lambda_{km}^2\hat{w}_k(\mathbf{x}) - \hat{d}_k(\mathbf{x})]\Omega_{km}(\mathbf{x}) = 0, \quad \mathbf{x}\in V \tag{6a}$$

$$\left[\alpha_k(\mathbf{x}) + \beta_k(\mathbf{x})\hat{K}_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]\Omega_{km}(\mathbf{x}) = 0, \quad \mathbf{x}\in S \tag{6b}$$

which results from the user choice of the simpler coefficients $\hat{K}_k(\mathbf{x})$, $\hat{w}_k(\mathbf{x})$, and $\hat{d}_k(\mathbf{x})$, and should then offer an exact analytical solution for the corresponding auxiliary eigenfunctions, while retaining information, as much as possible, from the original problem formulation. As a simplification limit, problem 6 can be taken as Helmholtz equations in the specific coordinates system of the considered problem. In general, the auxiliary problem retains the same boundary condition coefficients of the original eigenvalue problem, $\alpha_k(\mathbf{x})$ and $\beta_k(\mathbf{x})$, but in principle, even those could be simplified if required for achieving an analytical solution.

Problem 6 provides the integral transform pair below

$$\bar{\psi}_{km}^{(i)} = \int_V \hat{w}_k(\mathbf{x})\tilde{\Omega}_{km}(\mathbf{x})\psi_{ki}(\mathbf{x})dV, \quad \text{transform} \tag{7a}$$

$$\psi_{ki}(\mathbf{x}) = \sum_{m=1}^{\infty}\tilde{\Omega}_{km}(\mathbf{x})\bar{\psi}_{km}^{(i)}, \quad \text{inverse} \tag{7b}$$

where the normalized auxiliary eigenfunctions and corresponding norms are given by

$$\tilde{\Omega}_{km}(\mathbf{x}) = \frac{\Omega_{km}(\mathbf{x})}{\sqrt{N_{\Omega_{km}}}}, \quad \text{with} \tag{7c}$$

$$N_{\Omega_{km}} = \int_V \hat{w}_k(\mathbf{x})\Omega_{km}^2(\mathbf{x})dV \tag{7d}$$

Problem (2) is operated on with $\int_V\tilde{\Omega}_{km}(\mathbf{x})(.)dV$, to yield the transformed linear algebraic system, for each potential order k, truncated to the M_k^{th} order, written in matrix form as

$$(\mathbf{A}_k+\mathbf{C}_k)\{\bar{\boldsymbol{\psi}}_k\} = \boldsymbol{\mu}_k^2\mathbf{B}_k\{\bar{\boldsymbol{\psi}}_k\} \tag{8a}$$

with the elements of the $M_k\times M_k$ matrices and vector $\boldsymbol{\mu}_k$ given by [2,18]

$$\begin{aligned} a_{kmn} = &\int_S \frac{\tilde{\Omega}_{km}(\mathbf{x}) - \hat{K}_k(\mathbf{x})\dfrac{\partial\tilde{\Omega}_{km}(\mathbf{x})}{\partial\mathbf{n}}}{\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})} \\ &\left[\beta_k(\mathbf{x})\left(K_k(\mathbf{x})-\hat{K}_k(\mathbf{x})\right)\frac{\partial\tilde{\Omega}_{kn}(\mathbf{x})}{\partial\mathbf{n}}\right]dS \\ &-\int_S\left(K_k(\mathbf{x})-\hat{K}_k(\mathbf{x})\right)\tilde{\Omega}_{km}(\mathbf{x})\frac{\partial\tilde{\Omega}_{kn}(\mathbf{x})}{\partial\mathbf{n}}dS \\ &+\int_V\left(K_k(\mathbf{x})-\hat{K}_k(\mathbf{x})\right)\nabla\tilde{\Omega}_{km}(\mathbf{x})\cdot\nabla\tilde{\Omega}_{kn}(\mathbf{x})dV \\ &+\int_V\left(d_k(\mathbf{x})-\hat{d}_k(\mathbf{x})\right)\tilde{\Omega}_{km}(\mathbf{x})\tilde{\Omega}_{kn}(\mathbf{x})dV \end{aligned} \tag{8b}$$

$$b_{kmn} = \int_V w_k(\mathbf{x})\tilde{\Omega}_{km}(\mathbf{x})\tilde{\Omega}_{kn}(\mathbf{x})dV \tag{8c}$$

$$c_{kmn} = \lambda_{km}^2\delta_{mn} \tag{8d}$$

$$\boldsymbol{\mu}_k = \{\mu_{k1}, \mu_{k2}, \ldots, \mu_{kM_k}\}^T \tag{8e}$$

where δ_{mn} is the Kronecker delta. The choice of the truncation order M_k for the algebraic system (8) is directly associated with the truncation order (N_k) required for the transformed ODE system, Eq. (4), as will be discussed in what follows, since we should always have $M\geq N$. The constant coefficients' algebraic structure of Eq. (8*a*) makes it straightforward to implement a quick analytical iteration, starting from the diagonal form to the matrices, to inspect for convergence of the N_k first eigenvalues with the increase in M_k. It should be recalled that the first few eigenvalues are the dominant modes in the eigenfunction expansion, and the more restrictive accuracy control should be placed on them.

The above formal solutions based on the general Sturm–Liouville eigenfunction expansion base are quite effective and have been employed in the majority of applications handled through the GITT over the years, in combination with convergence acceleration techniques when required, such as single or multipass analytical explicit filtering, progressive filtering for multidimensional applications, implicit filtering for nonlinear problems, integral balance approach for a priori and a posteriori convergence enhancement, and infinite series convergence acceleration algorithms [16,43–46]. These possibilities act on rewriting the inverse formula (3*b*), whose convergence is in principle governed by the decaying behavior of the transformed potentials in absolute value with an increasing number of terms in the expansion. For instance, filtering proposals essentially extract information from the original source terms, to reduce their relative importance in the evolution of the transformed potentials both with increasing eigenvalue index and with the t variable, approaching the spectral convergence pattern typical of homogeneous linear problems. A general filtering proposal can be written as

$$T_k(\mathbf{x},t) = F_k(\mathbf{x},t) + T_k^*(\mathbf{x},t) \tag{9a}$$

where $F_k(\mathbf{x},t)$ are the proposed filters and $T_k^*(\mathbf{x},t)$ are the filtered potentials, given by

$$T_k^*(\mathbf{x},t) = \sum_{i=1}^{\infty}\tilde{\psi}_{ki}(\mathbf{x})\bar{T}_{k,i}^*(t) \tag{9b}$$

which is an eigenfunction expansion on the same base but with filtered transformed potentials.

This effect becomes clearer once we examine the formal solution of Eq. (5) after performing the integration by parts of the right-hand side as

$$\bar{T}_{ki}(t) = \bar{f}_{ki}\exp\left(-\mu_{ki}^2 t\right) + \frac{1}{\mu_{ki}^2}\left[\bar{g}_{ki}\left(t,\bar{\mathbf{T}}(t)\right) - \bar{g}_{ki}\left(0,\bar{\mathbf{T}}(0)\right)\exp\left(-\mu_{ki}^2 t\right)\right] - \frac{1}{\mu_{ki}^2}\int_0^t \frac{d\bar{g}_{ki}}{dt'}\exp\left[-\mu_{ki}^2(t-t')\right]dt' \tag{10}$$

From the second term in the r.h.s., it can be noticed that the transformed source term may dominate the convergence behavior, which would then be governed by the pattern $1/\mu_{ki}^2$, instead of the more favorable spectral pattern, $\exp(-\mu_{ki}^2 t)$. The formal solution in Eq. (10) is also quite useful in providing a first guess to the truncation order N_k of the transformed system, starting with the linearization of the source term and dropping out the third term in the r.h.s. This should be sufficient for an overall inspection of convergence rates but, if needed, an iterative analytical refinement [14] can be undertaken by considering the third term in Eq. (10) via symbolic computation.

Here, we do not attempt to review these various possibilities, but to discuss more closely another convergence enhancement path, which inherently brings more information on the original problem formulation into the eigenfunctions themselves. Whenever there are operators in the original problem that are not accounted for by the chosen Sturm–Liouville base, it intrinsically means that these missing operators are carried along into the source terms, either in the equations or boundary conditions, and end up by acting as modified sources that slow down the absolute decaying behavior of the transformed potentials. Therefore, either alone or in combination with other convergence acceleration schemes above discussed, Sec. 3 below reviews some previous work that, in the realm of the GITT, have considered nonclassical eigenvalue problems in the proposed eigenfunction expansions.

3 Nonclassical Eigenvalue Problems

The Sturm–Liouville theory is fully established and provides a reliable path for the proposition of eigenfunction expansions such as those illustrated in the above formal solutions. On the other hand, nonclassical eigenvalue problem choices, if not reducible through variables transformation to a Sturm–Liouville structure, in general, require the establishment of the corresponding orthogonality property and the spectral analysis to identify the possibility of occurrence of nonpositive real and/or complex eigenvalues in finite domains. Despite the added analytical difficulties, representing more closely the original operators of the proposed problem through a nonclassical eigenvalue problem proposal should lead to improvement in the expansion convergence, with the related reduction in computational costs, especially for nonlinear formulations.

The present review aims at providing a systematic presentation of different classes of problems where the employment of a nonclassical basis has resulted in representative gains in the integral transform approach. Due to space limitations, only a few of these extensions are analyzed more closely in Sec. 4.

3.1 Moving Boundary Problems. The first class of problems to be considered is associated with moving boundaries, such as phase change, ablation, oxidation, degradation, and other phenomena that alter the problem domain as the heat or mass transfer process evolves [16,28,29,47]. Even if the problem formulation fits the general representation in Eq. (1), the domain is now characterized by time-dependent external surface and overall volume, $S(t)$ and $V(t)$, respectively. Thus, the eigenvalues and eigenfunctions now become time-dependent and, except for simpler formulations when explicit expressions are obtainable for both, the eigenvalues need to be solved simultaneously with the transformed potentials. The time-dependent eigenvalue problem is given by

$$\nabla\cdot\left[K_k(\mathbf{x})\nabla\psi_{ki}(\mathbf{x},t)\right] + \left[\mu_{ki}^2(t)w_k(\mathbf{x}) - d_k(\mathbf{x})\right]\psi_{ki}(\mathbf{x},t) = 0, \quad \mathbf{x}\in V(t) \tag{11a}$$

$$\left[\alpha_k(\mathbf{x}) + \beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]\psi_{ki}(\mathbf{x},t) = 0, \quad \mathbf{x}\in S(t) \tag{11b}$$

resulting in the following integral transform pair

$$\bar{T}_{ki}(t) = \int_{V(t)} w_k(\mathbf{x})\tilde{\psi}_{ki}(\mathbf{x},t)T_k(\mathbf{x},t)\mathrm{d}V, \quad \text{transform} \tag{12a}$$

$$T_k(\mathbf{x},t) = \sum_{i=1}^{\infty}\tilde{\psi}_{ki}(\mathbf{x},t)\bar{T}_{ki}(t), \quad \text{inverse} \tag{12b}$$

where the normalized eigenfunction is defined as

$$\tilde{\psi}_{ki}(\mathbf{x},t) = \frac{\psi_{ki}(\mathbf{x},t)}{\sqrt{N_{ki}}} \tag{12c}$$

with

$$N_{ki}(t) = \int_{V(t)} w_k(\mathbf{x})\psi_{ki}(\mathbf{x},t)^2 dV \tag{12d}$$

A limited class of linear problems with a prescribed functional relation for the boundary movement have been proposed [14,17], but in general these moving boundary problems are inherently nonlinear and a heat or mass balance at the phase change interfaces closes the problem formulation and is not shown here, since these depend on the specific application. Due to the time-dependence of the eigenvalue problem, an additional term appears in the transformed system, in the form

$$\frac{d\bar{T}_{ki}(t)}{dt} + \sum_{j=1}^{\infty}A_{kij}(t)\,\bar{T}_{kj}(t) = \bar{g}_{ki}\left(t,\bar{\mathbf{T}}\right), t > 0, i, j = 1, 2\ldots \tag{13a}$$

where

$$A_{kij}(t) = \delta_{ij}\mu_{ki}^2(t) + \int_{V(t)} w_k(\mathbf{x})\frac{\partial\tilde{\psi}_{kj}(\mathbf{x},t)}{\partial t}\,\tilde{\psi}_{ki}(\mathbf{x},t)\,dV \tag{13b}$$

3.2 Heterogeneous Media. Class II problems as described in Ref. [9] involve the solution of linear diffusion problems in multiregion domains. Although the theory for finding the integral transform solution is well-established in this class, through the associated coupled multiregion eigenvalue problem with a single set of eigenvalues, dealing with a complex heterogeneous multidimensional region through this path can be cumbersome, both in the analytical and numerical tasks. A more straightforward alternative is reformulating the heterogeneous media problem into a single domain, rewriting it as a Class I problem with space variable coefficients and source terms that account for any sort of subregions transitions. This single domain strategy was introduced in the context of fluid-porous media instability problems [48,49] and conjugated heat transfer [50], and subsequently extended to different classes of heterogeneous media situations [32–36,51–56], including the consideration of interfacial resistances dealt with as fictitious layers [55,56].

Consider an extended Class II nonlinear diffusion problem defined in a heterogeneous media that is represented by n_V subregions with volumes $V_l = 1, 2, \ldots, n_V$, and the corresponding

potential and flux conditions at the interfaces. Different potentials are accounted for in each subregion, $T_{kl}(\mathbf{x},t)$, $k=1,2,\ldots,n$ and $l=1,2,\ldots,n_V$, including nonlinear equation and boundary source terms, respectively, $P_{kl}(\mathbf{x},t,\mathbf{T})$ and $\phi_{kl}(\mathbf{x},t,\mathbf{T})$, in the following formulation:

$$w_{kl}(\mathbf{x})\frac{\partial T_{kl}(\mathbf{x},t)}{\partial t}=\nabla\cdot\left[K_{kl}(\mathbf{x})\nabla T_{kl}(\mathbf{x},t)\right]-d_{kl}(\mathbf{x})T_{kl}(\mathbf{x},t)+P_{kl}(\mathbf{x},t,\mathbf{T}),\quad \mathbf{x}\in V_l,\ t>0,\ k=1,2,\ldots,n,\ l=1,2,\ldots,n_V \tag{14a}$$

with initial, interface, and boundary conditions given by

$$T_{kl}(\mathbf{x},0)=f_{kl}(\mathbf{x}),\quad \mathbf{x}\in V_l \tag{14b}$$

$$K_{kl}(\mathbf{x})\frac{\partial T_{kl}(\mathbf{x},t)}{\partial \mathbf{n}}=h_{klm}(\mathbf{x})[T_{kl}(\mathbf{x},t)-T_{km}(\mathbf{x},t)],\quad \mathbf{x}\in S_{lm},\ t>0 \tag{14c}$$

$$K_{kl}(\mathbf{x})\frac{\partial T_{kl}(\mathbf{x},t)}{\partial \mathbf{n}}=K_{km}(\mathbf{x})\frac{\partial T_{km}(\mathbf{x},t)}{\partial \mathbf{n}},\quad \mathbf{x}\in S_{lm},\ t>0 \tag{14d}$$

$$\left[\alpha_{kl}(\mathbf{x})+\beta_{kl}(\mathbf{x})K_{kl}(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}\right]T_{kl}(\mathbf{x},t)=\phi_{kl}(\mathbf{x},t,\mathbf{T}),\quad \mathbf{x}\in S_l,\ t>0 \tag{14e}$$

where $\mathbf{n}$ denotes the outward-drawn normal to the interfaces, S_{lm} represents interfacial surfaces, and S_l stands for external surfaces. The corresponding eigenvalue problem is then given by

$$\nabla\cdot[K_{kl}(\mathbf{x})\nabla\psi_{kli}(\mathbf{x})]+[\mu_{kl}^2 w_{kl}(\mathbf{x})-d_{kl}(\mathbf{x})]\psi_{kli}(\mathbf{x})=0,\quad \mathbf{x}\in V_l,\ k=1,2,\ldots,n,\ l=1,2,\ldots,n_V \tag{15a}$$

$$K_{kl}(\mathbf{x})\frac{\partial \psi_{kli}(\mathbf{x})}{\partial \mathbf{n}}=h_{klm}(\mathbf{x})[\psi_{kli}(\mathbf{x})-\psi_{kmi}(\mathbf{x})],\quad \mathbf{x}\in S_{lm},\ t>0 \tag{15b}$$

$$K_{kl}(\mathbf{x})\frac{\partial \psi_{kli}(\mathbf{x})}{\partial \mathbf{n}}=K_{km}(\mathbf{x})\frac{\partial \psi_{kmi}(\mathbf{x})}{\partial \mathbf{n}},\quad \mathbf{x}\in S_{lm},\ t>0 \tag{15c}$$

$$\left[\alpha_{kl}(\mathbf{x})+\beta_{kl}(\mathbf{x})K_{kl}(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}\right]\psi_{kli}(\mathbf{x})=0,\quad \mathbf{x}\in S_l,\ t>0 \tag{15d}$$

with orthogonality property given as

$$\sum_{l=1}^{n_V}\int_{V_l} w_{kl}(\mathbf{x})\psi_{kli}(\mathbf{x})\psi_{klj}(\mathbf{x})\mathrm{d}V=\delta_{ij}N_{ki} \tag{15e}$$

However, it is possible to rewrite problem (14) as one single domain with spatially variable coefficients and source terms, such as in Eq. (1), accounting for the material regions transitions, where the single domain and its bounding surface are written as

$$V=\sum_{l=1}^{n_V}V_l,\quad S=\sum_{l=1}^{n_V}S_l \tag{16}$$

Then, problem 2 can be adopted as the eigenvalue problem, with the corresponding equation and boundary condition coefficients associated with the single domain reformulation. To write the heterogeneous media problem, Eq. (14), and the related eigenvalue problem 15, as a single domain formulation that accounts for interface conditions with potential discontinuities, Eq. (14*c*), a fictitious thin layer can be introduced between the two subregions, with appropriate dimensionless thickness and conductivity, to reproduce the equivalent interface resistance [55,56].

3.3 Irregular Domains. Problem (1) and the corresponding eigenvalue problem, Eq. (2), are defined for any arbitrary region V. Therefore, the solution here reviewed is in principle valid for any irregular region and can be readily computed once a solution is offered to the eigenvalue problem and the volume and surface integrals in Eqs. (4*b*) and (4*c*) have been evaluated. Thus, in principle, if an analytical solution is available in this domain for the simplest form possible of the problem 6, known as the Helmholtz equation, this auxiliary problem would be readily employed in the expansion of the unknown eigenfunctions. There are more than eleven orthogonal coordinate systems for which separation of variables provides an exact solution to the Helmholtz equation [57], but these apply only to the corresponding geometries that can be mapped as fixed coordinates surface boundaries in such coordinate systems.

A general integral transform solution of multidimensional eigenvalue problems has been provided in Ref. [27] and applied in Refs. [58,59], not requiring its representation in a specific coordinate system that matches the boundary surfaces. The idea relies on a progressive integral transformation process based on one-dimensional auxiliary problems in each coordinate direction. Then, the irregular boundaries need to be mapped as functions of the coordinates in such a way that the progressive integral transformation scheme can be applied. For instance, in a three-dimensional rectangular coordinate system, the irregular region bounding surfaces could be represented as

$$x_0\le x\le x_1,\quad y_0(x)\le y\le y_1(x),\quad z_0(x,y)\le z\le z_1(x,y) \tag{17a}$$

Then, the volume integrals can be implemented in the appropriate order as

$$\int_{\mathrm{v}}(.)\mathrm{d}V=\int_{x_0}^{x_1}\int_{y_0(x)}^{y_1(x)}\int_{z_0(x,y)}^{z_1(x,y)}(.)dzdydx \tag{17b}$$

It should be recalled that the order of integration in Eq. (17*b*) in each independent variable becomes mandatory with the choice of irregular domain representation proposed in Eq. (17*a*), which in many situations may allow for analytical or at least semi-analytical integration. However, more general purely discrete numerical integration algorithms can be adopted in the overall region V, without requiring the boundaries mapping of Eq. (17*a*), allowing for the determination of the integral transformation coefficients in arbitrarily irregular domains.

Conformal mapping techniques have been applied in conjunction with the GITT approach [60], directly applied to the original partial differential problem, and in principle, the same idea could be employed in handling the associated auxiliary eigenvalue problem instead. A third solution path considers a fictitious domain that envelopes the original irregular region [61] and, through the appropriate definition of fictitious properties and boundary conditions, recovers the irregular eigenvalue problem solution. A comparative analysis of the methodologies employed in Refs. [58,59] and Ref. [61] for the solution of eigenvalue problems in irregular geometries with Dirichlet and Neumann boundary conditions was presented [61]. In such work, the former methodology was coined the coincident domain approach and the latter the fictitious domain approach. The results demonstrated that the coincident domain approach has a better performance for irregular boundaries with Dirichlet conditions, whereas the fictitious domain approach is better suited for Neumann conditions.

Also, the integral transformation of irregular domains through progressive one-dimensional transformations, such as described above, may be directly applied to the original partial differential problem, and was, in fact, the first proposed solution path for the application of the GITT in irregular domains [26], shown to be an interesting alternative in different applications [62–67].

3.4 Anisotropic Media. The analysis of linear diffusion problems in anisotropic media through integral transforms was first proposed in Ref. [22], as an extension of the classical integral

transform method, which accounts for general nonsymmetric anisotropy in finite regions. An adjoint eigenvalue problem is proposed that forms a biorthogonal set with the original one, which by itself would be nonself-adjoint. This analysis was later expanded and written more concisely through a convenient matrix operator form [23]. This approach was then further generalized, employing the GITT, to handle nonlinear and heterogeneous media problems [68,69].

Therefore, if in Eq. (1*a*), the diffusion operator is characterized by the tensorial coefficient, $K_{kij}(\mathbf{x})$, instead of a scalar one, the eigenvalue problem obtained through separation of variables would then be given by

$$L_k\psi_k(\mathbf{x}) = \mu_k^2 w_k(\mathbf{x})\psi_k(\mathbf{x}), \quad \text{in } \mathbf{x} \in V \tag{18a}$$

with boundary conditions given as

$$B_k\psi_k(\mathbf{x}) = 0, \quad \mathbf{x} \in S \tag{18b}$$

with the operators being given as

$$L_k \equiv -\frac{\partial}{\partial x_i}\left(K_{kij}(\mathbf{x})\frac{\partial}{\partial x_j}\right) + d_k(\mathbf{x}) \tag{18c}$$

$$B_k \equiv \alpha_k(\mathbf{x}) + \beta_k(\mathbf{x})\, n_i\, K_{kij}(\mathbf{x})\frac{\partial}{\partial x_j} \tag{18d}$$

For a general anisotropy tensor, the eigenvalue problem 18 is nonself-adjoint and the resulting eigenfunctions are not orthogonal. Following Refs. [22,23,70], an adjoint eigenvalue problem is proposed, which will lead to orthogonal eigenfunctions to those from Eq. (18), as

$$L_k^*\psi_k^*(\mathbf{x}) = \mu_k^2 w_k(\mathbf{x})\psi_k^*(\mathbf{x}), \quad \text{in } \mathbf{x} \in V \tag{19a}$$

with boundary conditions given as

$$B_k^*\psi_k^*(\mathbf{x}) = 0, \quad \mathbf{x} \in S \tag{19b}$$

where the adjoint operators are given by

$$L_k^* \equiv -\frac{\partial}{\partial x_i}\left[K_{kji}(\mathbf{x})\frac{\partial}{\partial x_j}\right] + d_k(\mathbf{x}) \tag{19c}$$

$$B_k^* \equiv \alpha_k(\mathbf{x}) + \beta_k(\mathbf{x})\, n_i\, K_{kji}(\mathbf{x})\frac{\partial}{\partial x_j} \tag{19d}$$

Then, the eigenfunctions $\psi_k(\mathbf{x})$ and $\psi_k^*(\mathbf{x})$ obey the orthogonality condition below, yielding a biorthogonal set

$$\int_V w_k(\mathbf{x})\psi_{kn}(\mathbf{x})\psi_{km}^*(\mathbf{x})dV = \delta_{nm}N_{kn} \tag{20a}$$

where δ_{nm} is the Kronecker delta and the norms are given by

$$N_{kn} = \int_V w_k(\mathbf{x})\psi_{kn}(\mathbf{x})\psi_{kn}^*(\mathbf{x})dV \tag{20b}$$

The integral transform-inverse pair is then established with the help of the orthogonality property, as

$$\bar{T}_{kn}(t) = \int_V w_k(\mathbf{x})T_k(\mathbf{x},t)\psi_{kn}^*(\mathbf{x})dV \quad \text{transform} \tag{21a}$$

$$T_k(\mathbf{x},t) = \sum_{n=1}^{\infty}\frac{1}{N_{kn}}\psi_{kn}(\mathbf{x})\bar{T}_{kn}(t) \quad \text{inverse} \tag{21b}$$

The GITT approach itself can be employed in transforming the original differential eigenvalue problems into algebraic ones, by proposing a simpler auxiliary eigenvalue problem to provide a base for the eigenfunction expansions, such as previously described. A similar path of considering a biorthogonal expansion may also be followed in solving the eigenvalue problem, for enhanced convergence also in finding eigenvalues and eigenfunctions.

3.5 Coupled Equations. Class IV problems in Ref. [9] correspond to linear diffusion problems of any arbitrary number of potentials that are coupled through source (or sink) terms in the governing balance equations. The exact integral transform solution of this class of problems was originally proposed in Ref. [69], by considering the corresponding coupled set of eigenvalue problems. In Ref. [69], the potentials are symmetrically coupled through linear source terms, and the coupling can be split from the source terms of Eq. (1*a*), as

$$P_k(\mathbf{x},t,\mathbf{T}) = \hat{P}_k(\mathbf{x},t,\mathbf{T}) + b(\mathbf{x})\sum_{p=1}^{n}\sigma_{kp}[T_p(\mathbf{x},t) - T_k(\mathbf{x},t)] \tag{22a}$$

$$\sigma_{kp} = \sigma_{pk} \tag{22b}$$

Then, the integral transform solution is constructed through the following coupled eigenvalue problem [9,69]:

$$\begin{aligned}&\nabla\cdot[K_k(\mathbf{x})\nabla\psi_k(\mathbf{x})] + [\mu^2 w_k(\mathbf{x}) - d_k(\mathbf{x})]\psi_k(\mathbf{x})\\ &\quad + b(\mathbf{x})\sum_{p=1}^{n}\sigma_{kp}[\psi_p(\mathbf{x}) - \psi_k(\mathbf{x})]\\ &\quad = 0, \ \mathbf{x} \in V\end{aligned} \tag{23a}$$

$$\left[\alpha_k(\mathbf{x}) + \beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}\right]\psi_k(\mathbf{x}) = 0, \quad \mathbf{x} \in S \tag{23b}$$

while the orthogonality property of the eigenfunctions is given by

$$\sum_{k=1}^{n}\int_V w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\psi_{kj}(\mathbf{x})\mathrm{d}V = \delta_{ij}N_i \tag{23c}$$

where the norms are computed from

$$N_i = \sum_{k=1}^{n}\int_V w_k(\mathbf{x})\psi_{ki}^2(\mathbf{x})\mathrm{d}V \tag{23d}$$

The transform-inverse pair is deduced as [9,69]

$$\bar{T}_i(t) = \sum_{k=1}^{n}\int_V w_k(\mathbf{x})\psi_{ki}(\mathbf{x})T_k(\mathbf{x},t)\mathrm{d}V, \quad \text{transform} \tag{24a}$$

$$T_k(\mathbf{x},t) = \sum_{i=1}^{\infty}\frac{1}{N_i}\psi_{ki}(\mathbf{x})\bar{T}_i(t), \quad \text{inverse} \tag{24b}$$

This choice of eigenfunction expansion base was recently extended to the analysis of more complex nonlinear problems in heterogeneous media [70,71]. As anticipated, substantial gains in the analytical and computational steps of the methodology were achieved. The coupled eigenvalue problems enable the integral transformation process to merge all the spatial and coupling information into a single ordinary differential system for the transformed potentials. Since the information on the coupling of the different potentials is carried on into the coupled eigenvalue problems, even though just through characteristic linear functional forms, improved convergence rates are achieved in comparison to the most usual integral transformation path with uncoupled Sturm–Liouville problems. The nonsymmetric coupling

formulation was also considered in Ref. [72], which is coined as Class V problems in Ref. [9], though an exact solution through integral transforms was reported only for the case of two coupled potentials.

Coupled eigenvalue problems also commonly occur in linear stability analyses of heat and fluid flow problems. These problems have been solved using the GITT itself by employing a basis stemming from simpler eigenvalue problems that admit analytical solutions, as described in Refs. [48,49,73–75]. The consideration of coupled auxiliary eigenvalue problems opens out an interesting alternative for enhancing convergence in such applications.

3.6 Coupled Boundary Conditions. Class III problems in reference [9] involve two linear diffusion equations, such as Eq. (1) but with linear source terms, that are coupled through the boundary conditions in a more general form, thus leading to coupled eigenvalue problems to achieve an exact solution through integral transforms [76]. The Class III eigenvalue problem is written as

$$\nabla\cdot[K_k(\mathbf{x})\nabla\psi_k(\mathbf{x})]+[\mu^2 w_k(\mathbf{x})-d_k(\mathbf{x})]\psi_k(\mathbf{x})=0,\quad k=1,2\quad \mathbf{x}\in V \tag{25a}$$

$$B_{k1}\psi_1(\mathbf{x})+B_{k2}\psi_2(\mathbf{x})=0,\quad \mathbf{x}\in S_1,\quad k=1,2 \tag{25b}$$

$$B_k\psi_k(\mathbf{x})=0,\quad \mathbf{x}\in S_2,\quad k=1,2 \tag{25c}$$

$$B_k=\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right] \tag{25d}$$

$$B_{km}=\left[\alpha_{km}(\mathbf{x})+\beta_{km}(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right],\quad k,m=1,2 \tag{25e}$$

while the orthogonality property of the eigenfunctions is given by

$$\sum_{k=1}^{2}\sigma_k\int_V w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\psi_{kj}(\mathbf{x})\mathrm{d}V=\delta_{ij}N_i \tag{26a}$$

where the norms are computed from

$$N_i=\sum_{k=1}^{2}\sigma_k\int_V w_k(\mathbf{x})\psi_{ki}^2(\mathbf{x})\mathrm{d}V \tag{26b}$$

with the σ_k coefficients obtained from the determinants below

$$\sigma_1=\det\begin{bmatrix}\beta_{11} & \alpha_{11}\\ \beta_{21} & \alpha_{21}\end{bmatrix} \tag{26c}$$

$$\sigma_2=\det\begin{bmatrix}\alpha_{12} & \beta_{12}\\ \alpha_{22} & \beta_{22}\end{bmatrix} \tag{26d}$$

The transform-inverse pair is deduced as [9,76]

$$\bar{T}_i(t)=\sum_{k=1}^{2}\sigma_k\int_V w_k(\mathbf{x})\psi_{ki}(\mathbf{x})T_k(\mathbf{x},t)\mathrm{d}V,\quad \text{transform} \tag{27a}$$

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}\frac{1}{N_i}\psi_{ki}(\mathbf{x})\bar{T}_i(t),\quad \text{inverse} \tag{27b}$$

This class of diffusion problems was initially motivated by the analysis of drying in moist capillary porous media [77] and coupled heat and mass transfer in concurrent channel flow [78]. Concerning the solution of Luikov equations of drying, it was soon observed in Ref. [79] that the eigenvalue problem 25 could lead to complex eigenvalues, which if not accounted for could lead to erroneous results in the temperature and moisture distributions. In addition, the numerical computation of such complex eigenvalues should be undertaken with care, since missing just one of these roots could again affect the results. Considering this difficulty, the GITT was employed on this system of equations but employing the alternative path of considering a simpler decoupled eigenvalue problems set [80]. This solution path was also extended to the solution of nonlinear drying problems [81] and, more recently, to the analysis of intense drying [82], including the pressure field as a third dependent variable. Nevertheless, the GITT itself, as applied to the solution of this eigenvalue problem coupled at the boundary conditions, following the same basic ideas as in Eqs. (6)–(8), is a powerful tool in automatically determining real and complex eigenvalues through the corresponding algebraic matrix eigensystem solution [83], thus directly applying the exact solution of Class III problems.

Eigenvalue problems such as Eq. (25) were also solved in the analysis of the hydrodynamically developed flow of a gas in a duct whose walls are coated with a sublimating material [78] and in thermally developing flows in concurrent flow within double-pipe heat exchangers [84].

3.7 Boundary Conditions With Finite Capacitance. We now consider diffusion problems such as in Eq. (1), but with a more general boundary condition type, involving a thin boundary film with finite capacitance. The corresponding boundary condition is written as

$$\gamma_k(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial t}+\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]T_k(\mathbf{x},t)=\phi_k(\mathbf{x},t,\mathbf{T}),\quad \mathbf{x}\in S,\quad t>0 \tag{28}$$

where the boundary capacitance coefficient, $\gamma_k(\mathbf{x})$, becomes zero wherever there are no capacitance effects at the boundary surface S.

The integral transform analysis of the linear version of this class of problems was introduced in Ref. [21]. Applying the separation of variables to the homogeneous version of the original problem, the nonclassical eigenvalue problem is given as [21]

$$\nabla\cdot[K_k(\mathbf{x})\nabla\psi_k(\mathbf{x})]+[\mu_k^2 w_k(\mathbf{x})-d_k(\mathbf{x})]\psi_k(\mathbf{x})=0,\quad \mathbf{x}\in V \tag{29a}$$

$$\mu_k^2\gamma_k(\mathbf{x})\psi_k(\mathbf{x})=\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial\mathbf{n}}\right]\psi_k(\mathbf{x}),\quad \mathbf{x}\in S \tag{29b}$$

with the squared eigenvalue appearing also in the boundary condition, while the orthogonality property of the eigenfunctions is given by

$$\int_V w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\psi_{kj}(\mathbf{x})\mathrm{d}V+\int_S\frac{\gamma_k(\mathbf{x})}{\beta_k(\mathbf{x})}\psi_{ki}(\mathbf{x})\psi_{kj}(\mathbf{x})\mathrm{d}S=\delta_{ij}N_{ki} \tag{29c}$$

where the norms are computed from

$$N_{ki}=\int_V w_k(\mathbf{x})\psi_{ki}^2(\mathbf{x})\mathrm{d}V+\int_S\frac{\gamma_k(\mathbf{x})}{\beta_k(\mathbf{x})}\psi_{ki}^2(\mathbf{x})\mathrm{d}S \tag{29d}$$

The transform-inverse pair is deduced as [21]

$$\bar{T}_{ki}(t)=\int_V w_k(\mathbf{x})\psi_{ki}(\mathbf{x})T_k(\mathbf{x},t)\mathrm{d}V+\int_S\frac{\gamma_k(\mathbf{x})}{\beta_k(\mathbf{x})}\psi_{ki}(\mathbf{x})T_k(\mathbf{x},t)\mathrm{d}S,\quad \text{transform} \tag{30a}$$

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}\frac{1}{N_{ki}}\psi_{ki}(\mathbf{x})\bar{T}_{ki}(t),\quad \text{inverse} \tag{30b}$$

In heat transfer problems, boundary condition (28) is usually called a fifth kind boundary condition and, for $\alpha_k(\mathbf{x})=0$, it is called a fourth kind boundary condition. An interesting application to the transient thermal analysis of a nuclear fuel rod, including the ceramic fuel and a lumped formulation for the cladding, is presented in Ref. [85]. There, however, the GITT was employed to avoid the above eigenvalue problem, and a simpler Sturm–Liouville problem was employed, such as in Eq. (2). In such situations, since the original problem and the chosen eigenvalue problem do not present the same operators in the boundary conditions, to avoid slower convergence of the eigenfunction expansions, either an implicit filtering [47,81,86,87] or an integral balance approach [16,29,43,45] should be adopted.

In Ref. [88], the GITT was applied to transient three-dimensional pumping of aquifers, considering a fully penetrating vertical well between two parallel streams. Different physical situations were considered, but the unconfined aquifer case introduces a fifth-kind boundary condition, which is partially handled as described above. A simpler three-dimensional eigenvalue problem that is uncoupled in the three coordinates is considered instead, though accounting for the finite capacitance in the corresponding coordinate, allowing for the straightforward definition of the three-dimensional eigenfunction expansion.

3.8 Boundary Conditions With Unknown Source Term. A more general coupling of diffusion equations through the boundary conditions is represented by the so-called Class VI problems in Ref. [9], when there is a gross substance balance along the boundary surface, that does not allow for a priori knowledge of the boundary condition source terms, ϕ_k, in Eq. (1c), which then becomes part of the solution. This class of problems was first motivated by the extraction of substances from a solid porous medium [89], and the exact integral transform solution was obtained by combining the Laplace transform and the classical integral transform methods, based on a nonclassical eigenvalue problem. In the proposed general formulation, each diffusion equation (1a) can be defined in a specific region V_k, and the boundary conditions are now given as [9]

$$\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}\right]T_k(\mathbf{x},t)=\phi(t),\quad \mathbf{x}\in S_k,\ t>0 \tag{31a}$$

where the unknown source term, $\phi(t)$, is coupled to the potentials through the overall substance balance as

$$\frac{d\phi(t)}{dt}+\sum_{k=1}^{n}\gamma_k\int_{S_k}K_k(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial \mathbf{n}}dS=Q(t),\quad t>0 \tag{31b}$$

with initial condition

$$\phi(t)=\phi_0 \tag{31c}$$

After employing the Laplace transform method and applying separation of variables to the homogeneous version of the original problem, the resulting nonclassical eigenvalue problem is given as [9,89]

$$\nabla\cdot[K_k(\mathbf{x})\nabla\psi_k(\mathbf{x})]+[\mu^2 w_k(\mathbf{x})-d_k(\mathbf{x})]\psi_k(\mathbf{x})=0,\quad \mathbf{x}\in V_k,\quad k=1,2,\ldots n \tag{32a}$$

$$\sum_{k=1}^{n}\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_k(\mathbf{x})dV+\left[\alpha_k(\mathbf{x})+\beta_k(\mathbf{x})K_k(\mathbf{x})\frac{\partial}{\partial \mathbf{n}}\right]\psi_k(\mathbf{x})=0,\mathbf{x}\in S_k \tag{32b}$$

with the orthogonality property of the eigenfunctions given by

$$\sum_{k=1}^{n}\alpha_k\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\psi_{kj}(\mathbf{x})\mathrm{d}V+\left(\sum_{k=1}^{n}\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\mathrm{d}V\right)\left(\sum_{k=1}^{n}\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{kj}(\mathbf{x})\mathrm{d}V\right)=\delta_{ij}N_i \tag{32c}$$

where the norms are computed from

$$N_i=\sum_{k=1}^{n}\alpha_k\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{ki}^2(\mathbf{x})\mathrm{d}V+\left[\sum_{k=1}^{n}\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\mathrm{d}V\right]^2 \tag{32d}$$

The transform-inverse pair is deduced as [9]

$$\tilde{\bar{T}}_i(s)=\sum_{k=1}^{n}\alpha_k\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\tilde{T}_k(\mathbf{x},s)\mathrm{d}V+\left(\sum_{k=1}^{n}\gamma_k\int_{V_k}w_k(\mathbf{x})\psi_{ki}(\mathbf{x})\mathrm{d}V\right)\left(\sum_{k=1}^{n}\gamma_k\int_{V_k}w_k(\mathbf{x})\tilde{T}_k(\mathbf{x},s)\mathrm{d}V\right),$$

transform

(33a)

$$\tilde{T}_k(\mathbf{x},s)=\sum_{i=1}^{\infty}\frac{1}{N_i}\psi_{ki}(\mathbf{x})\tilde{\bar{T}}_{ki}(s),\quad \text{inverse} \tag{33b}$$

where s is the Laplace transform variable and the Laplace transformed potential is defined as

$$\tilde{T}_k(\mathbf{x},s)=\int_0^{\infty}e^{-st}T_k(\mathbf{x},t)dt \tag{33c}$$

An important class of problems in dealing with double-pipe heat exchangers follows the same basic formulation, leading to the same eigenvalue problem, once one of the streams is lumped and the thermal resistance due to the wall is accounted for [90–92]. Then, the convection–diffusion equation for one of the streams is coupled through the boundary condition with the lumped temperature of the other stream. Here, the longitudinal coordinate plays the role of the time variable. The GITT was the solution methodology in these works, but a simpler eigenvalue problem was chosen, coupling the transformed temperatures with the lumped stream temperature. Again in Ref. [85], on transient thermal analysis of a nuclear fuel rod, but for the model that also accounts for the gap thermal resistance, the lumped cladding temperature becomes an unknown boundary quantity, such as described in Eq. (31), and must be simultaneously determined with the temperature distribution in the ceramic pellet, following a heat balance that is mathematically analogous to Eq. (31b). In this case, the GITT was employed with a simpler Sturm–Liouville problem, thus coupling the transformed transient ceramic temperatures with a priori unknown cladding temperature. More recently [93], an integral transform solution of the concurrent parallel plates moving bed heat exchanger was proposed, when the solid and fluid temperatures are coupled such as described above. In this case, the solution of the actual nonclassical eigenvalue problem was pursued, to find the proper exact solution.

3.9 Eigenvalue Problem With Nonlinear Dependence of the Eigenvalue. A classical problem in convective heat transfer is known as the extended Gräetz problem, due to the consideration of axial heat diffusion in the fluid. The exact analytical solution of this problem via separation of variables [24] leads to an eigenvalue problem where coefficients appear with nonlinear dependence on the eigenvalues; more specifically in this case, the eigenvalue appears both squared and to the fourth power in the same equation. A similar class of problems was analyzed in a different application [94]. The GITT itself, with a simpler Sturm–Liouville eigenvalue problem, was employed in obtaining approximate analytical solutions to this extended Graetz problem [95], by considering only the diagonal terms in the coefficients matrix and was later utilized to obtain hybrid numerical–analytical solutions of the complete coupled system

[9,96], as well as semi-analytical solutions to the complete system in infinite channels [97]. This same solution path of GITT adopting a simpler Sturm–Liouville problem was also employed in handling steady and transient conjugated heat transfer problems with axial diffusion effects [51,52,98].

In Ref. [25], the GITT itself was employed in solving this class of eigenvalue problems with nonlinear dependence of the eigenvalue, following the same strategy described in Eqs. (6–8), then leading to the exact integral transform solution of the initially posed problem. The nonclassical eigenvalue problem with nonlinear dependence on the eigenvalue is written in this case as

$$\nabla \cdot [K_k(\mathbf{x}^*)\nabla\psi_{ki}(\mathbf{x}^*)] + [\mu_{ki}^4 q_k(\mathbf{x}^*) + \mu_{ki}^2 w_k(\mathbf{x}^*) - d_k(\mathbf{x}^*)]\psi_{ki}(\mathbf{x}^*) = 0, \quad \mathbf{x}^* \in V^* \tag{34a}$$

$$\left[\alpha_k(\mathbf{x}^*) + \beta_k(\mathbf{x}^*)K_k(\mathbf{x}^*)\frac{\partial}{\partial \mathbf{n}}\right]\psi_{ki}(\mathbf{x}^*) = 0, \quad \mathbf{x}^* \in S^* \tag{34b}$$

where V^* corresponds at most to a two-dimensional region, with the corresponding position vector $\mathbf{x}^*$, since the elimination of the axial variable is already accomplished, which causes the appearance of the nonlinear dependence on the eigenvalue. This nonclassical eigenfunction expansion provides a significant convergence enhancement effect and was also employed in the analysis of conjugated heat transfer problems with and without slip flow conditions [51,52,55,56,98].

3.10 Convective Eigenvalue Problems. The GITT solution of convection–diffusion problems dates to Refs. [99–102], when a purely diffusive eigenvalue problem has been adopted, such as in Eq. (2), while the convective terms are incorporated into the source terms in Eq. (1*a*). Though this approach has been quite successful in different classes of applications in transport phenomena, for highly convective situations, it remains of interest to employ convective eigenvalue problems that can provide a convergence enhancement effect. For the linear constant coefficients' situation, a dependent variable transformation has been proposed in Ref. [103] that eliminates the convection terms and collapses the information into new equation coefficients. In combination with the integral transform method, analytical solutions were obtained for three-dimensional linear convection–diffusion problems in both transient and steady-states. The eigenfunction expansion of the modified problem resulted in improved convergence in comparison with the GITT solution directly applied to the original problem, employing the simple diffusive eigenvalue problem.

The dependent variable transformation pathway was extended to handle nonlinear convection–diffusion problems in Refs. [34,35,104], through consideration of the original variable velocity fields in the transformation, thus leading to a modified diffusive nonself-adjoint eigenvalue problem, or by considering a characteristic, yet simple, representation of the velocity fields that could still allow for obtaining a modified self-adjoint eigenvalue problem, thus mixing the variable transformation with the GITT application.

Consider the modified version of Eq. (1*a*), for a single potential for simplicity, but now explicitly incorporating the convective term in its left-hand side, as

$$w(\mathbf{x})\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla\cdot[K(\mathbf{x})\nabla T] - d(\mathbf{x})T(\mathbf{x},t) + P(\mathbf{x},t,T), \quad \mathbf{x}\in V \tag{35}$$

Instead of merging the convective term into the source term, $P(\mathbf{x},t,T)$, and proceeding with the traditional GITT as previously described, one may first propose the dependent variable transformation to obtain a modified diffusion problem formulation. For the sake of illustration, vector $\mathbf{u}$ is represented by the three components $\{u_x, u_y, u_z\}$ for a general three-dimensional situation, in the Cartesian coordinates system, $\mathbf{x} = \{x, y, z\}$. Equation (35) is then given in the generalized diffusive form as [34,104]

$$\hat{w}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = \hat{k}_y(\mathbf{x})\hat{k}_z(\mathbf{x})\frac{\partial}{\partial x}\left[\hat{k}_x(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x}\right] + \hat{k}_x(\mathbf{x})\hat{k}_z(\mathbf{x})\frac{\partial}{\partial y}\left[\hat{k}_y(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial y}\right] + \hat{k}_x(\mathbf{x})\hat{k}_y(\mathbf{x})\frac{\partial}{\partial z}\left[\hat{k}_z(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial z}\right] - \hat{d}(\mathbf{x})T(\mathbf{x},t) + \hat{P}(\mathbf{x},t,T), \quad \mathbf{x}\in V, \quad t>0 \tag{36a}$$

where

$$\hat{K}(\mathbf{x}) = \hat{k}_x(\mathbf{x})\hat{k}_y(\mathbf{x})\hat{k}_z(\mathbf{x}) \tag{36b}$$

$$\hat{w}(\mathbf{x}) = w(\mathbf{x})\,\hat{K}(\mathbf{x})/K(\mathbf{x}) \tag{36c}$$

$$\hat{d}(\mathbf{x}) = d(\mathbf{x})\,\hat{K}(\mathbf{x})/K(\mathbf{x}) \tag{36d}$$

$$\hat{P}(\mathbf{x},t,T) = P(\mathbf{x},t,T)\,\hat{K}(\mathbf{x})/K(\mathbf{x}) \tag{36e}$$

$$\mathbf{u}^*(\mathbf{x}) = \frac{1}{K(\mathbf{x})}[\mathbf{u}(\mathbf{x}) - \nabla K(\mathbf{x})] \tag{36f}$$

$$\hat{k}_x(\mathbf{x}) = e^{-\int u_x^*(\mathbf{x})dx} \tag{36g}$$

$$\hat{k}_y(\mathbf{x}) = e^{-\int u_y^*(\mathbf{x})dy} \tag{36h}$$

$$\hat{k}_z(\mathbf{x}) = e^{-\int u_z^*(\mathbf{x})dz} \tag{36i}$$

By imposing adequate restrictions on the choices of the characteristic linear coefficients $K(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$, the transformed diffusion coefficients shall be given as functions of only the corresponding space coordinate, or $\hat{k}_x(\mathbf{x}) = \hat{k}_x(x)$, $\hat{k}_y(\mathbf{x}) = \hat{k}_y(y)$, $\hat{k}_z(\mathbf{x}) = \hat{k}_z(z)$, and then the resulting generalized diffusion formulation will lead to a classical self-adjoint eigenvalue problem [34,104], such as in Eqs. (2).

These previous contributions have in common that advective and diffusive terms are partially or fully merged into a generalized diffusion problem. The goal is to offer an eigenfunction expansion base that partially incorporates the convective effects, toward expansions with improved convergence rates. The transformation strategy allows for different choices of reference coefficients, which lead to different effects on overall convergence enhancement.

3.11 Biharmonic Eigenvalue Problems. The integral transform analysis of either the boundary layer or the Navier–Stokes equations has been introduced under both the primitive variables [105–108] and stream function-only (for two-dimensional problems) or scalar/vector potentials (for three-dimensional problems) formulations [30,31,109–113]. The stream function formulation has been clearly preferred over the years, due to the improved convergence behavior, automatic satisfaction of the continuity equation, and inherent elimination of the pressure field, as reviewed in Ref. [20]. The associated eigenvalue problem employed in combination with the GITT is a fourth-order one, like the biharmonic eigenvalue problems commonly used in vibration problems [114]. As a matter of fact, the GITT approach with biharmonic eigenvalue problems has been increasingly employed in different classes of problems in structural dynamics, such as detailed in Refs. [115–120].

More recently, a novel vector eigenfunction expansion proposal [36] has unified the GITT solution of the two- and three-

dimensional primitive variables formulations, by collapsing in one set of transformed potentials, all velocity components. The velocity vector field representation is physically interpreted as the influence of an infinite number of vortices disturbing a base flow. The stream function-only formulation for the two-dimensional situation is then automatically recovered as a special case. This approach has been advanced in recent extensions dealing with fluid flow and heat or mass transfer within channels and cavities filled with fluids or partially filled with a saturated porous medium [36,37,53,54].

Thus, due to its marked differences from Eq. (1), consider the transient Navier–Stokes equations in dimensionless vector form for an incompressible flow, given as [37]

$$\nabla \cdot \boldsymbol{u} = 0, \quad \boldsymbol{x} \in V \tag{37a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \frac{1}{\mathrm{Re}} \nabla \cdot [\mu \nabla \boldsymbol{u}], \quad \boldsymbol{x} \in V \tag{37b}$$

where V represents the fluid domain, $\boldsymbol{u}$ the dimensionless velocity vector, p the dimensionless pressure field, Re the Reynolds number, and μ the dimensionless variable viscosity. The associated eigenvalue problem for the GITT application is obtained for the linear case of Stoke's flow, when $Re \to 0$. The associated self-adjoint eigenvalue problem equation is written as [37]

$$\nabla \times \nabla \cdot [\mu \nabla (\nabla \times \tilde{\boldsymbol{\Phi}}_i)] + \mu_i^2 (\nabla \times \nabla \times \tilde{\boldsymbol{\Phi}}_i) = \boldsymbol{0} \tag{38}$$

Following the proposition of separating the flow field into a base flow and an infinite number of vortices, and after applying a filter to the velocity field, the integral transform pair is proposed in terms of the base vector [37]

$$\hat{\boldsymbol{u}}(\mathbf{x}, t) = \sum_{i=1}^{\infty} \bar{u}_i(t) (\nabla \times \tilde{\boldsymbol{\Phi}}_i), \quad \text{inverse} \tag{39a}$$

$$\bar{u}_i(t) = \int_V (\nabla \times \tilde{\boldsymbol{\Phi}}_i) \cdot \hat{\boldsymbol{u}}(\mathbf{x}, t) dV, \quad \text{transform} \tag{39b}$$

where the filtered velocity field, $\hat{\boldsymbol{u}}(\boldsymbol{x}, t)$, comes from

$$\boldsymbol{u}(\mathbf{x}, t) = \hat{\boldsymbol{u}}(\mathbf{x}, t) + \boldsymbol{u}_f(\mathbf{x}, t) \tag{40}$$

3.12 Nonlinear Eigenvalue Problems. Except for those moving boundary problems when the domain evolution is not known a priori, and must be determined as part of the solution, all the alternative nonclassical eigenvalue problems considered so far do not account for nonlinearities in the equation or boundary coefficients. Essentially, the nonlinearities in the original formulation are merged into the source terms, $P_k(\mathbf{x}, t, \mathbf{T})$ and $\phi_k(\mathbf{x}, t, \mathbf{T})$. Then, the eigenvalue problems are in general constructed with characteristic linear functional forms of the equations and boundary conditions coefficients, $w_k(\mathbf{x})$, $K_k(\mathbf{x})$, $d_k(\mathbf{x})$, $\alpha_k(\mathbf{x})$, and $\beta_k(\mathbf{x})$, which are not necessarily identical to the coefficients in the original problem formulation. For this reason, all the nonlinearities of the problem are concentrated into the transformed system, and thus, on the transformed potential.

The idea of carrying along the nonlinearities in the coefficients of the original problem anticipates the perception that eigenvalues and eigenfunctions shall now be functions of the actual potentials that they are supposed to represent, thus inherently requiring some sort of coupled solution of potentials and eigenfunctions. Such as in moving boundary problems, eigenvalues, and consequently eigenfunctions and norms, become functions of the potentials that are now solved simultaneously. This concept was formally introduced in Refs. [38,121], with the GITT working with a nonlinear eigenfunction expansion base due to a nonlinear boundary condition in a heat conduction problem, as reviewed in [3]. Thus, consider a diffusion problem such as given by Eq. (1), but for a single potential and with nonlinear boundary condition coefficients, written as

$$\alpha(\mathbf{x}, t, T)\, T + \beta(\mathbf{x}, t, T)\, K(\mathbf{x}) \frac{\partial T}{\partial \mathbf{n}} = \phi(\mathbf{x}, t, T), \quad \mathbf{x} \in S, \; t > 0 \tag{41}$$

where α and β are the nonlinear boundary condition coefficients.

An implicit filter can be introduced to account for the nonlinear boundary source term an offer a homogeneous boundary condition for the filtered potential, $T^*(\mathbf{x},t)$. Then, the nonlinear eigenvalue problem is adopted, which still carries the original nonlinear boundary condition coefficients, in the form

$$\nabla . K(\mathbf{x}) \nabla \psi_i(\mathbf{x}; t) + [\mu_i^2(t) w(\mathbf{x}) - d(\mathbf{x})] \psi_i(\mathbf{x}; t) = 0, \quad \mathbf{x} \in \mathrm{V} \tag{42a}$$

with boundary conditions

$$\alpha(\mathbf{x}, t, T) \psi_i(\mathbf{x}; t) + \beta(\mathbf{x}, t, T) \mathrm{K}(\mathbf{x}) \frac{\partial \psi_i(\mathbf{x}; t)}{\partial \mathbf{n}} = 0, \quad \mathbf{x} \in \mathrm{S} \tag{42b}$$

and the associated time-dependent eigenfunctions, $\psi_i(\mathbf{x}; t)$, and eigenvalues, $\mu_i(t)$, are for the moment assumed as known. Problem (42), through the corresponding orthogonality property, allows for the development of the following integral transform pair:

$$\bar{T}_i(t) = \int_V w(\mathbf{x})\, \psi_i(\mathbf{x}; t)\, T^*(\mathbf{x}, t)\, dV, \quad \text{transform} \tag{43a}$$

$$T^*(\mathbf{x}, t) = \sum_{i=1}^{\infty} \frac{1}{N_i(t)} \psi_i(\mathbf{x}; t) \bar{T}_i(t), \quad \text{inverse} \tag{43b}$$

with norms given by

$$N_i(t) = \int_V w(\mathbf{x}) \psi_i^2(\mathbf{x}; t)\, dV \tag{43c}$$

Other applications on heat and mass transfer have already been reported in the literature adopting nonlinear eigenfunction expansions, including coupled nonlinear problems and the combination of nonlinear boundary conditions, and moving boundary problems [39,47,122].

4 Applications

Four application problems are now considered more closely, that offer each an opportunity of convergence enhancement through one of the nonclassical eigenvalue problem choices here reviewed. This analysis serves as both illustration of the convergence improvement and confirmation of the achievable accuracy, under each proposed eigenfunction expansion base.

4.1 Transient Heat or Mass Diffusion in Heterogeneous Slab. Let us consider a heterogeneous slab consisting of n different components, with concentrations or temperatures given by $T_k(x, t)$, $k = 1,2,\ldots,n$, which are continuous functions of space and time in the region V defined by $x \in [0, L]$. This is an example from Ref. [9] that corresponds to the present Sec. 3.5 on Coupled Equations, with the coupling term introduced by Eqs. (22*a*) and (22*b*), thus modifying Eq. (1*a*). The problem formulation, such as for a three-component chemically reacting mass diffusion problem in a slab, is written in dimensionless form as [9]

$$w_k \frac{\partial T_k(R, t)}{\partial t} = K_k \frac{\partial^2 T_k(R, t)}{\partial R^2} + \sum_{p=1}^{3} \sigma_{kp} [T_p(R, t) - T_k(R, t)],$$
$$\sigma_{kp} = \sigma_{pk}, \quad R \in [0, 1], \quad k = 1, 2, 3, \quad t > 0 \tag{44a}$$

with initial and boundary conditions given by

$$T_k(R,0) = 1, \qquad R \in [0,1] \tag{44b}$$

$$\frac{\partial T_k(R,t)}{\partial R} = 0, \quad R = 0,\ t > 0 \tag{44c}$$

$$T_k(R,t) = 0, \quad R = 1,\ t > 0 \tag{44d}$$

The corresponding nonclassical coupled eigenvalue problem is then given by Eqs. (23*a*) and (23*b*), simplified as

$$K_k \frac{d^2\psi_k(R)}{dR^2} + \mu^2 w_k \psi_k(R) + \sum_{p=1}^{3} \sigma_{kp}\left[\psi_p(R) - \psi_k(R)\right] = 0, \tag{45a}$$

$$R \in [0,1], \quad k = 1,2,3$$

$$\frac{d\psi_k(R)}{dR} = 0, \quad R = 0 \tag{45b}$$

$$\psi_k(R) = 0, \quad R = 1 \tag{45c}$$

The exact integral transform solution has also been verified with the direct application of the GITT with a simpler decoupled Sturm–Liouville eigenvalue problem, which does not include the coupling term in Eq. (44*a*), written as

$$K_k \frac{d^2\Omega_k(R)}{dR^2} + \lambda_k^2 w_k \Omega_k(R) = 0, \qquad R \in [0,1],\ k = 1,2,3 \tag{46a}$$

$$\frac{d\Omega_k(R)}{dR} = 0, \quad R = 0 \tag{46b}$$

$$\Omega_k(R) = 0, \quad R = 1 \tag{46c}$$

Numerical results are here reported for the following combination of parameters: $K_k = (0.1,\ 1,\ 10)$, $w_k = (1,\ 1,\ 1)$, and three different cases in terms of the coupling coefficients value, $\sigma_{kp} = \sigma = (0,\ 1,\ 10)$, where the case $\sigma = 0$ corresponds to the absence of coupling, for checking purposes.

Tables 1 and 2 provide a brief convergence analysis of the integral transform solution for the coupled potentials, $T_k(R,t)$, $k = 1,2,3$, at $R = 0$ and $t = 0.02$ and 0.1, respectively, employing the nonclassical eigenvalue problem proposal. As usual, the convergence of eigenfunction expansions is slowed down as the time variable t gets smaller. Nevertheless, full convergence to five significant digits is achieved for all three potentials for truncation orders as low as $N < 20$. The numerical results for the GITT solution with decoupled Sturm–Liouville problems are not shown here since it agrees to all five significant digits. In fact, for this linear problem, the analytical solution would be the same through the two paths, but the algorithm can slightly affect the overall precision and, most importantly, the computational cost. When adopting the decoupled eigenvalue problem, the integral transformation process eventually leads to a coupled ordinary differential system for the transformed potentials, which once solved numerically raises the computational cost and introduces some numerical error due to the methodology employed in handling the initial value problem. In the present linear problem, only the sixth digit of the potentials was affected by choosing the Sturm–Liouville base, but the computational cost was noticeably increased. Also shown, at the last line of each table, are the numerical results obtained through the Method of Lines as implemented in the *Mathematica* routine NDSolve [40], employing variable finite difference order and variable step size. At the smaller time t, results from the analytical and numerical approaches are in better agreement, but some noticeable deviation is observed in the numerical solution for larger times, but still agreeing to the third significant digit with the fully converged integral transform solutions.

Table 1 Convergence of the coupled potentials $T_k(0,t)$, $k = 1,2,3$, at $t = 0.02$ and $\sigma = 10$ in chemically reacting mass diffusion problem in a heterogeneous slab

N	$k=1$	$k=2$	$k=3$
2	0.87793	0.93125	0.81837
4	0.94723	0.98306	0.82139
6	0.97048	0.98663	0.82161
8	0.98074	0.98704	0.82165
10	0.98536	0.98714	0.82166
12	0.98727	0.98716	0.82166
14	0.98798	0.98717	0.82166
16	0.98820	0.98717	0.82166
18	0.98827	0.98717	0.82166
20	0.98828	0.98717	0.82166
NDSolve [40]	0.98830	0.98718	0.82194

Table 2 Convergence of the coupled potentials $T_k(0,t)$, $k = 1,2,3$, at $t = 0.1$ and $\sigma = 10$ in chemically reacting mass diffusion problem in a heterogeneous slab

N	$k=1$	$k=2$	$k=3$
2	0.70135	0.69107	0.39463
4	0.71872	0.69496	0.39506
6	0.72078	0.69509	0.39507
8	0.72095	0.69510	0.39507
10	0.72096	0.69510	0.39507
12	0.72096	0.69510	0.39507
14	0.72096	0.69510	0.39507
16	0.72096	0.69510	0.39507
18	0.72096	0.69510	0.39507
20	0.72096	0.69510	0.39507
NDSolve [40]	0.72134	0.69552	0.39578

Figure 1 compares the results for the three potentials evolution, $T_k(R,t)$, $k = 1,2,3$, at $R = 0$, with $\sigma = 10$ and 1, respectively. Although not noticeable to the graph scale, the solutions with GITT and a decoupled Sturm–Liouville problem and with the numerical routine NDSolve [40] are also plotted together, with different line thicknesses, and overall excellent adherence. As the value of σ increases, and thus the coupling effect, the curves for the three potentials tend to merge.

It is also of interest to examine the behavior of the dimensionless wall mass flux (at R = 1) along with the time variable. Figure 2 provides the evolution of the three potential fluxes, represented by the derivative $\partial T_k(1,t)/\partial R$ plotted for the case $\sigma = 10$ and $N = 30$ terms.

The dimensionless fluxes are fully converged to at least five significant digits, with the slowest convergence observed at $t = 0.01$ for the first potential ($k = 1$). The results from the default use of the NDSolve function in [40] are also plotted, in thiner lines, where it can be seen from the deviation of the solid thicker (GITT) and thiner (NDSolve) lines, that the Method of Lines shall require user prescribed mesh refinement to reach full agreement with the fully converged GITT results for lower values of t, especially for $k = 1$. For instance, the five digits converged GITT results for the flux at $t = 0.01$, for the three potentials, is given by $\{-15.666,\ -5.6187,\ -1.9994\}$, while NDSolve provides $\{-18.880,\ -5.6036,\ -1.9978\}$, clearly requiring a mesh convergence analysis for lower values of t. Restricting the maximum step sizes of the spatial and temporal variables to, respectively, 0.002 and 0.01, the Method of Lines [40] results change to $\{-15.517,\ -5.5646,\ -1.9799\}$, which at least for the critical case of $k = 1$ markedly improves, recovering the agreement to the graphical scale of the two sets of results.

4.2 Line Heat Source Method for Thermophysical Properties of Liquids. The line heat source method is a well-established experimental technique for the determination of thermal

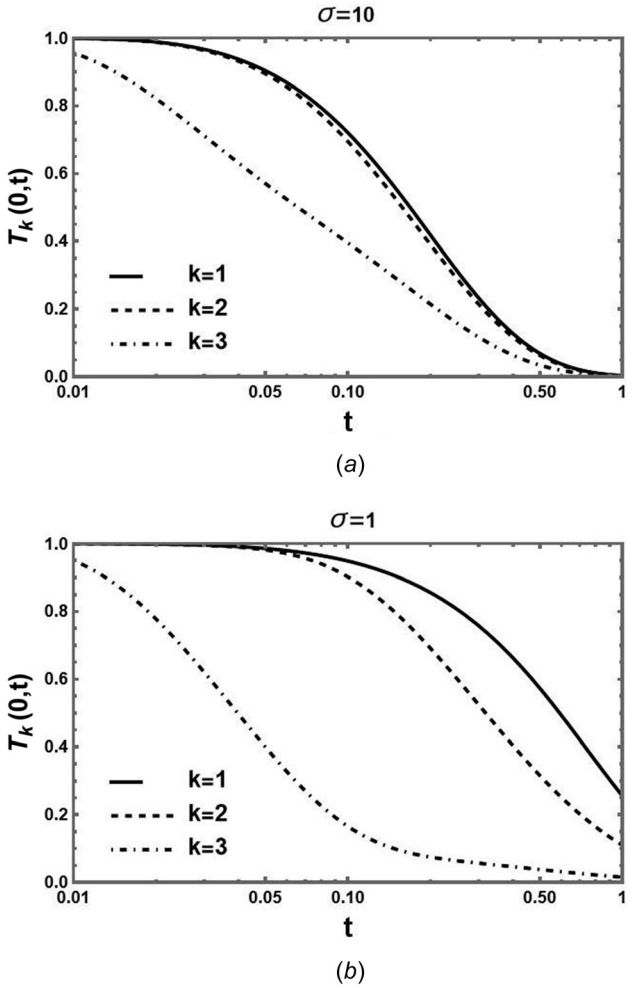

Fig. 1 Evolution of the three potentials concentration in chemically reacting mass diffusion problem in a heterogeneous slab, $T_k(0, t)$, $k = 1, 2, 3$, with (*a*) $\sigma = 10$ and (*b*) $\sigma = 1$, for all three methodologies (CITT, GITT, and method of lines)

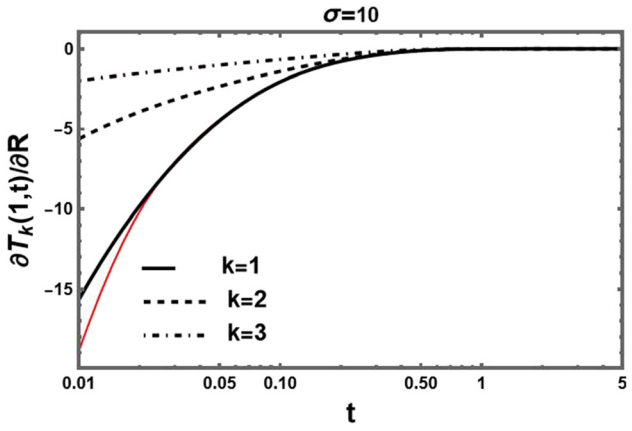

Fig. 2 Evolution of the three potentials dimensionless wall mass fluxes in chemically reacting mass diffusion problem in a heterogeneous slab, $\partial T_k(1, t)/\partial R$, $k = 1,2,3$, with $\sigma = 10$, for GITT in black lines, and Method of Lines [40] in thiner lines

diffusivity and conductivity of liquids [123,124]. It consists of a probe that encapsulates an electrical resistance and thermocouples, immersed in a cylindrical reservoir of the liquid with unknown thermophysical properties. After a power pulse that heats the probe, the thermocouple registers the transient temperature evolution at the interface with the fluid, from which the properties are estimated. A reduced mathematical model can be employed for the experimental design, which helps define the reservoir volume, the probe power source, and the duration of the pulse, among other parameters. Here, a reduced model is proposed that lumps the temperature distributions in both the probe body and the reservoir bounding wall, while keeping a distributed formulation within the fluid region. It then results in a formulation for the fluid temperature that incorporates finite capacitances at both boundaries of the cylindrical region, in the form

$$\frac{1}{\alpha_f}\frac{\partial T_f(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_f(r,t)}{\partial r}\right], \qquad r_s < r < r_i, \quad t > 0 \tag{47a}$$

$$T_f(r,0) = T_{f,0} \tag{47b}$$

$$-k_f\frac{\partial T_f(r,t)}{\partial r}\bigg|_{r_s} = \frac{r_s}{2}g_0 - \rho_s cp_s\frac{r_s}{2}\frac{\partial T_f(r,t)}{\partial t}\bigg|_{r_s}, \quad t > 0 \tag{47c}$$

$$k_f\frac{\partial T_f(r,t)}{\partial r}\bigg|_{r_i} = \frac{r_e}{r_i}h\left[T_\infty - T_f(r_i,t)\right] - \frac{\rho_m cp_m\left(r_e^2 - r_i^2\right)}{2r_i}\frac{\partial T_f(r,t)}{\partial t}\bigg|_{r_i}, \quad t > 0 \tag{47d}$$

where r_s, r_i, and r_e represent, respectively, the probe radius, the internal and the external radii of the reservoir wall; k, ρc_p, and α represent, respectively, the thermal conductivity, volumetric thermal capacity, and thermal diffusivity of each material, while the indices f, m, and s refer to the fluid, the reservoir wall, and the probe, respectively; g_0, h, and T_∞ represent the volumetric heat generation rate within the probe, the effective heat transfer coefficient of the wall with the external ambient, and the external ambient temperature.

Equation (47) belongs to the class of problems discussed in Sec. 3.7, which refers to boundary conditions with finite capacitance. The first step in the integral transform solution is the proposition of a filtering solution to eliminate the nonhomogeneities in Eqs. (47*c*) and (47*d*), which is obtained from the steady-state solution of problem (47), or

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_{f,F}(r)}{\partial r}\right] = 0, \qquad r_s < r < r_i \tag{48a}$$

$$-k_f\frac{\partial T_{f,F}(r)}{\partial r}\bigg|_{r_s} = \frac{r_s}{2}g_0 \tag{48b}$$

$$k_f\frac{\partial T_{f,F}(r)}{\partial r}\bigg|_{r_i} = \frac{r_e}{r_i}h[T_\infty - T_{f,F}(r_i,t)] \tag{48c}$$

Thus, applying the filtering $T_f(r,t) = T_f^*(r,t) + T_{f,F}(r)$, the filtered problem becomes

$$\frac{1}{\alpha_f}\frac{\partial T_f^*(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_f^*(r,t)}{\partial r}\right], \qquad r_s < r < r_i, \quad t > 0 \tag{49a}$$

$$T_f^*(r,0) = f(r) = T_{f,0} - T_{f,F}(r) \tag{49b}$$

$$-k_f\frac{\partial T_f^*(r,t)}{\partial r}\bigg|_{r_s} = -\rho_s cp_s\frac{r_s}{2}\frac{\partial T_f^*(r,t)}{\partial t}\bigg|_{r_s}, \quad t > 0 \tag{49c}$$

$$-k_f\frac{\partial T_f^*(r,t)}{\partial r}\bigg|_{r_i} = \frac{r_e}{r_i}hT_f^*(r_i,t) + \frac{\rho_m cp_m\left(r_e^2 - r_i^2\right)}{2r_i}\frac{\partial T_f^*(r,t)}{\partial t}\bigg|_{r_i}, \quad t > 0 \tag{49d}$$

The nonclassical eigenvalue problem that leads to the exact solution is given by

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\psi(r)}{dr}\right]+\mu^2\psi(r)=0,\qquad r_s<r<r_i \tag{50a}$$

$$-k_f\frac{d\psi(r)}{dr}\bigg|_{r_s}=\rho_s cp_s\frac{r_s}{2}\alpha_f\mu^2\psi(r_s) \tag{50b}$$

$$-k_f\frac{d\psi(r)}{dr}\bigg|_{r_i}=\left[\frac{r_e}{r_i}h-\frac{\rho_m cp_m\left(r_e^2-r_i^2\right)}{2r_i}\alpha_f\mu^2\right]\psi(r_i) \tag{50c}$$

with normalization integral

$$N_j=\int_{r_s}^{r_i}r\psi_j^2(r)dr+r_s\frac{\rho_s cp_s}{\rho_f cp_f}\frac{r_s}{2}\psi_j^2(r_s)+r_i\frac{\rho_m cp_m}{\rho_f cp_f}\frac{\left(r_e^2-r_i^2\right)}{2r_i}\psi_j^2(r_i) \tag{50d}$$

The exact integral transform solution is then written as

$$T_f(r,t)=T_{f,F}(r)+\sum_{j=1}^{\infty}\frac{1}{N_j}\psi_j(r)\bar{f}_j e^{-\mu_j^2 t} \tag{51a}$$

where $\bar{f}_j$ is the transformed initial condition, given as

$$\bar{f}_j=\int_{r_s}^{r_i}r\psi_j(r)f(r)dr+r_s\frac{\rho_s cp_s}{\rho_f cp_f}\frac{r_s}{2}\psi_j(r_s)f(r_s)+r_i\frac{\rho_m cp_m}{\rho_f cp_f}\frac{\left(r_e^2-r_i^2\right)}{2r_i}\psi_j(r_i)f(r_i) \tag{51b}$$

where the eigenfunctions, $\psi_j(r)$, and eigenvalues, μ_j, are obtained from

$$\psi_j(r)=S_jJ_0(\mu_j r)-V_jY_0(\mu_j r) \tag{51c}$$

$$S_jU_j-V_jW_j=0 \tag{51d}$$

with

$$S_j=-\mu_jY_1(\mu_j r_i)\left[\frac{r_e}{r_i}\frac{h}{k_f}-\frac{\rho_m cp_m}{\rho_f cp_f}\frac{\left(r_e^2-r_i^2\right)}{2r_i}\mu_j^2\right]Y_0(\mu_j r_i) \tag{51e}$$

$$V_j=-\mu_jJ_1(\mu_j r_i)\left[\frac{r_e}{r_i}\frac{h}{k_f}-\frac{\rho_m cp_m}{\rho_f cp_f}\frac{\left(r_e^2-r_i^2\right)}{2r_i}\mu_j^2\right]J_0(\mu_j r_i) \tag{51f}$$

$$W_j=-\mu_jY_1(\mu_j r_s)\left(\frac{\rho_s cp_s}{\rho_f cp_f}\frac{r_s}{2}\mu_j^2\right)Y_0(\mu_j r_s) \tag{51g}$$

$$U_j=-\mu_jJ_1(\mu_j r_s)\left(\frac{\rho_s cp_s}{\rho_f cp_f}\frac{r_s}{2}\mu_j^2\right)J_0(\mu_j r_s) \tag{51h}$$

The integral transform solution of Eqs. (47) through the GITT with a classical Sturm–Liouville problem would normally be undertaken by dropping out the terms involving the squared eigenvalue in the boundary conditions, Eqs. (50*b*) and (50*c*). However, since the boundary conditions for the eigenfunctions would then not satisfy the filtered problem boundary conditions, Eqs. (49*c*) and (49*d*), convergence could become very slow, especially in the vicinity of these positions, and would certainly not converge at the boundary position itself. Then, an integral balance scheme applied over Eq. (49*a*) would be required or an implicit filter should be used, coupling the filter and the transformed potentials, to improve convergence.

Numerical results are here reported for a test case involving the physical properties and dimensions from an actual experimental setup [123,124], involving a stainless steel probe, agar gel, and a Pyrex reservoir, as: $r_s=0.6$ mm, $r_i=4.3$ cm, $r_e=4.491$ cm, $\rho_s=7790$ kg/m^3, $c_{ps}=470$ J/kg °C, $\rho_m=2200$ kg/m^3, $c_{pm}=750$ J/kg °C, $\rho_f=1000$ kg/m^3, $c_{pf}=4180$ J/kg °C, $k_f=0.6$ W/m °C, $g_0=1.21261\times10^7$ W/m^3, $h=20$ W/m^2 °C, $T_{f,0}=26.9$ °C, $T_\infty=25.4$ °C. The heat pulse in the probe is analyzed for $t=15$, 30, 60, and 120 s. Tables 3 and 4 provide a brief convergence analysis for the fluid temperature at different radial positions, at $t=15$ and 60 s, respectively. The two extreme radial positions correspond to the interface with the probe (r_i) and with the reservoir wall (r_e), and the intermediate position is 1 mm distant from the heating probe. Truncation orders up to $N=30$ have been considered, and it can clearly be inspected that full convergence to six significant digits has been achieved in all positions and for both time values, as expected with a faster convergence rate for the larger value of t. Comparative numerical results for the NDSolve routine [40] could not be directly obtained, since its algorithm does not deal with time derivatives in the boundary conditions. In any case, for verification purposes, the GITT results for the time derivatives in the boundary conditions were fed into the NDSolve problem formulation as prescribed boundary source terms, and then the Method of Lines could be implemented and reproduced to the graph scale the fully converged temperature distributions here obtained.

Table 3 Convergence of the fluid temperature (°C) in the line heat source problem at different radial positions for $t=15$ s

N	$r=r_s=0.06$ cm	$r=0.16$ cm	$r=r_i=4.3$ cm
3	35.2762	31.7424	26.6060
6	33.0414	29.6094	26.8636
9	32.2652	28.9374	26.8280
12	31.9537	28.7072	26.8464
15	31.8563	28.6510	26.8429
18	31.8275	28.6397	26.8442
21	31.8210	28.6384	26.8440
24	31.8197	28.6385	26.8440
27	31.8195	28.6385	26.8440
30	31.8195	28.6385	26.8440

Table 4 Convergence of the fluid temperature (°C) in the line heat source problem at different radial positions for $t=60$ s

N	$r=r_s=0.06$ cm	$r=0.16$ cm	$r=r_i=4.3$ cm
3	35.6734	32.1360	26.6090
6	34.3680	30.8849	26.7762
9	34.2154	30.7513	26.7664
12	34.2032	30.7421	26.7673
15	34.2028	30.7418	26.7673
18	34.2028	30.7418	26.7673
21	34.2028	30.7418	26.7673
24	34.2028	30.7418	26.7673
27	34.2028	30.7418	26.7673
30	34.2028	30.7418	26.7673

Figure 3 provides the radial distribution of the fluid temperature within the reservoir, for the four different time values ($t=15$, 30, 60, and 120 s). Also shown are the initial temperature in the fluid and the external environment temperature. It can be observed that the slight temperature profile distortion close to the external reservoir wall is essentially due to the temperature difference between the cell initial temperature, $T_{f,0}$, and the external environment T_∞. Thus, the boundary condition at the reservoir external wall does not have a significant effect on the solution within the medium, especially close to the probe region, and should not influence the thermophysical properties measurements, considering the relatively short duration of the heat pulses and sufficiently large reservoir radius, as required by this technique. Also shown in this figure are the profiles obtained with the routine NDSolve [40] but

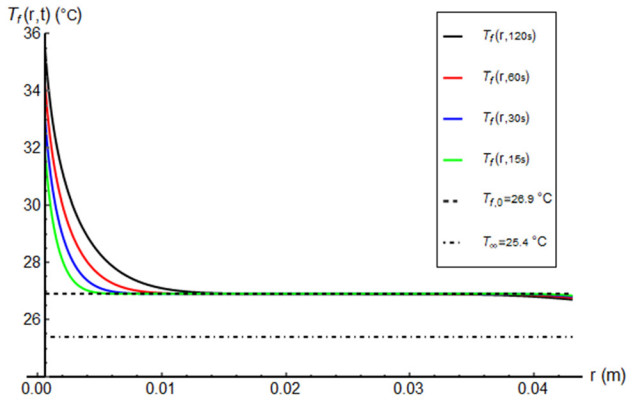

Fig. 3 Radial distribution of the fluid temperatures (°C) in the line heat source problem for different time values ($t = 15$, 30, 60, and 120 s)

employing the GITT solution to estimate the time derivatives in the boundary conditions, Eqs. (49*c*) and (49*d*), which have excellent adherence to the graph scale.

The convergence analysis for the heat flux at the probe surface ($r = r_s = 0.06$ cm) is provided in Table 5, for different time values, $t = 15$, 30, and 60 s. The truncation orders are extended to $N = 36$ just to reconfirm the convergence of the heat flux for the lowest time value. It can be observed that full convergence to six significant digits is achieved for truncation orders of $N \leq 30$ in all three cases, with more terms being required, as usual in eigenfunction expansions, at shorter times. The heat flux at the boundary can either be computed from the direct differentiation of the temperature expansion with respect to the spatial variable, r, or from the boundary condition, Eq. (47*c*), by differentiating the inverse formula with respect to t. As expected, both results were verified to be identical.

4.3 Local Thermal Non-Equilibrium Formulation for Convective Heat Transfer in Partially Porous Channels. Convective heat transfer in porous media is usually handled through the local thermal equilibrium (LTE) hypothesis, essentially stating that, at the pore level, the temperature difference between the solid and fluid phases is negligible. Such assumption stems from the small length scales typical of porous materials and the associated improvement on heat transfer. However, there are situations in which this hypothesis ceases to hold, most notably when the difference between the solid and fluid phases thermal conductivities is large 125, which is usually the case in heat transfer equipment design. For these cases, employment of local thermal nonequilibrium (LTNE) formulations is inevitable, i.e., assuming there is a finite heat transfer resistance at the pore level between the fluid and solid phases, yielding a coupled two-equation model

Table 5 Convergence of the heat flux at the probe surface (W/m^2) at different times $t = 15$, 30, and 60 s ($r = r_s = 0.06$ cm)

N	$t = 15$ s	$t = 30$ s	$t = 60$ s
3	3627.58	3627.96	3628.66
6	3596.87	3603.22	3612.29
9	3564.83	3585.33	3606.45
12	3539.17	3577.10	3605.52
15	3525.64	3575.07	3605.47
18	3519.62	3574.72	3605.47
21	3517.71	3574.69	3605.47
24	3517.20	3574.69	3605.47
27	3517.09	3574.69	3605.47
30	3517.08	3574.69	3605.47
33	3517.07	3574.69	3605.47
36	3517.07	3574.69	3605.47

in fluid-saturated porous domains. Figure 4 depicts a parallel plates channel partially filled with a fluid-saturated porous medium. A more detailed description of the model and the hypothesis involved can be found in Ref. [71].

Here, we present only the dimensionless form of the LTNE convective heat transfer model for the fluid and solid phases, written as

$$0 = k_{s,\text{eff}}\left(\frac{1}{\text{Pe}^2}\frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2}\right) - a_{fs}Nu_{fs}(T_s - T_{f1}), \qquad 0 \leq y \leq y_p \tag{52a}$$

$$u_p\frac{\partial T_{f1}}{\partial x} = k_{f,\text{eff}}\left(\frac{1}{\text{Pe}^2}\frac{\partial^2 T_{f1}}{\partial x^2} + \frac{\partial^2 T_{f1}}{\partial y^2}\right) - a_{fs}Nu_{fs}(T_{f1} - T_s), \qquad 0 \leq y \leq y_p \tag{52b}$$

$$u_f\frac{\partial T_{f2}}{\partial x} = k_f\left(\frac{1}{\text{Pe}^2}\frac{\partial^2 T_{f2}}{\partial x^2} + \frac{\partial^2 T_{f2}}{\partial y^2}\right), \qquad y_p < y \leq 1 \tag{52c}$$

with boundary conditions given by

$$T_s(0, y) = 1 \tag{52d}$$

$$T_{f1}(0, y) = 1 \tag{52e}$$

$$T_{f2}(0, y) = 1 \tag{52f}$$

$$\left.\frac{\partial T_s}{\partial x}\right|_{x\to\infty} = \left.\frac{\partial T_{f1}}{\partial x}\right|_{x\to\infty} = \left.\frac{\partial T_{f2}}{\partial x}\right|_{x\to\infty} = 0 \tag{52g-52i}$$

$$k_{f,\text{eff}}\left.\frac{\partial T_{f1}}{\partial y}\right|_{y=y_p} = \left.\frac{\partial T_{f2}}{\partial y}\right|_{y=y_p} - Nu_{\text{int}}\left[T_{f1}(x, y_p) - T_s(x, y_p)\right] \tag{52j}$$

$$k_{s,\text{eff}}\left.\frac{\partial T_s}{\partial y}\right|_{y=y_p} = \text{Nu}_{\text{int}}\left[T_{f1}(x, y_p) - T_s(x, y_p)\right] \tag{52k}$$

$$T_{f1}(x, y_p) = T_{f2}(x, y_p) \tag{52l}$$

$$T_s(x, 0) = T_{f1}(x, 0) = 0; \tag{52m}$$

$$\left.\frac{\partial T_{f2}}{\partial y}\right|_{y=1} = 0 \tag{52o}$$

This configuration brings a new challenge, method-wise, when compared with Class IV problems described in Sec. 3.5. Instead of coupled potentials defined within coincident domains, in Fig. 4 and Eqs. (52*a*)–(52*o*), the fluid phase domain extrapolates the porous medium region. Recently [71], an analytical solution was obtained by extending the eigenvalue problem of Eqs. (23*a*)–(23*d*) to noncoincident domains of validity of coupled potentials. This eigenvalue problem, applied to the physical situation of Fig. 4, is given by

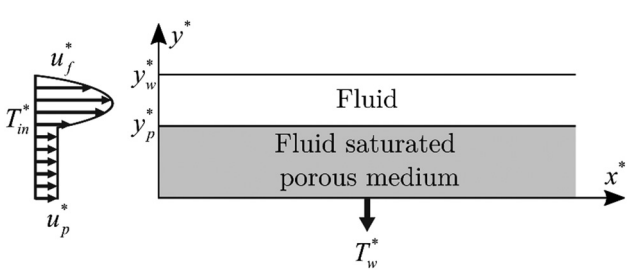

Fig. 4 Parallel plates channel partially filled with a fluid-saturated porous medium, with uniform temperature at the bottom wall and adiabatic top wall

$$k_{s,\mathrm{eff}}\frac{d^2\psi_{si}}{dy^2}+\lambda_i^2 k_{s,\mathrm{eff}}\psi_{si}-a_{fs}Nu_{fs}\left(\psi_{si}-\psi_{f1i}\right)=0,\qquad 0\le y\le y_p \tag{53a}$$

$$k_{f,\mathrm{eff}}\frac{d^2\psi_{f1i}}{dy^2}+\lambda_i^2 k_{f,\mathrm{eff}}\psi_{f1i}-a_{fs}Nu_{fs}\left(\psi_{f1i}-\psi_{si}\right)=0,\qquad 0\le y\le y_p \tag{53b}$$

$$\frac{d^2\psi_{f2i}}{dy^2}+\lambda_i^2\psi_{f2i}=0,\qquad y_p<y\le 1 \tag{53c}$$

with boundary conditions given as

$$k_{f,\mathrm{eff}}\frac{d\psi_{f1i}}{dy}\bigg|_{y=y_p}=\frac{d\psi_{f2i}}{dy}\bigg|_{y=y_p}-Nu_{\mathrm{int}}\left[\psi_{f1i}(y_p)-\psi_{si}(y_p)\right] \tag{53d}$$

$$k_{s,\mathrm{eff}}\frac{d\psi_{si}}{dy}\bigg|_{y=y_p}=Nu_{\mathrm{int}}\left[\psi_{f1i}(y_p)-\psi_{si}(y_p)\right] \tag{53e}$$

$$\psi_{f1i}(y_p)=\psi_{f2i}(y_p) \tag{53f}$$

$$\psi_{si}(0)=\psi_{f1i}(0)=0; \tag{53g}$$

$$\frac{d\psi_{f2i}}{dy}\bigg|_{y=1}=0 \tag{53i}$$

while the normalization integral and normalized eigenfunctions are written in the form

$$N_i=\int_0^{y_p}\left[k_{s,\mathrm{eff}}\psi_{si}^2(y)+k_{f,\mathrm{eff}}\psi_{f1i}^2(y)\right]\mathrm{d}y+\int_{y_p}^1\psi_{f2i}^2(y)\mathrm{d}y \tag{53j}$$

$$\tilde{\psi}_{si}(y)=\frac{\psi_{si}(y)}{\sqrt{N_i}} \tag{53k}$$

$$\tilde{\psi}_{f1i}(y)=\frac{\psi_{f1i}(y)}{\sqrt{N_i}} \tag{53l}$$

$$\tilde{\psi}_{f2i}(y)=\frac{\psi_{f2i}(y)}{\sqrt{N_i}} \tag{53m}$$

and bearing the following orthogonality property [71]

$$\int_0^{y_p}\left[k_{s,\mathrm{eff}}\tilde{\psi}_{si}(y)\tilde{\psi}_{sj}(y)+k_{f,\mathrm{eff}}\tilde{\psi}_{f1i}(y)\tilde{\psi}_{f1j}(y)\right]\mathrm{d}y+\int_{y_p}^1\tilde{\psi}_{f2i}(y)\tilde{\psi}_{f2j}(y)\mathrm{d}y=\delta_{ij} \tag{53n}$$

The orthogonality property of Eq. (53*n*) allows for the proposition of transform-inverse pairs for each potential as follows:

$$\bar{T}_i(x)=\int_0^{y_p}\left[k_{s,\mathrm{eff}}\tilde{\psi}_{si}(y)T_s(x,y)+k_{f,\mathrm{eff}}\tilde{\psi}_{f1i}(y)T_{f1}(x,y)\right]\mathrm{d}y+\int_{y_p}^1\tilde{\psi}_{f2i}(y)T_{f2}(x,y)\mathrm{d}y \tag{54a}$$

$$T_s(x,y)=\sum_{i=1}^{\infty}\tilde{\psi}_{si}(y)\bar{T}_i(x) \tag{54b}$$

$$T_{f1}(x,y)=\sum_{i=1}^{\infty}\tilde{\psi}_{f1i}(y)\bar{T}_i(x) \tag{54c}$$

$$T_{f2}(x,y)=\sum_{i=1}^{\infty}\tilde{\psi}_{f2i}(y)\bar{T}_i(x) \tag{54d}$$

Table 6 Convergence of the bulk fluid temperature and the Nusselt number at different axial positions when the local thermal nonequilibrium hypothesis is adopted

	$x=0.0001$		$x=0.0002$		$x=0.0003$	
M	T_{bf}	Nu	T_{bf}	Nu	T_{bf}	Nu
5	0.885634	8.09353	0.838438	6.75854	0.799184	6.11414
15	0.887049	8.53914	0.838415	6.84102	0.798952	6.12042
25	0.887053	8.54196	0.838415	6.84101	0.798952	6.12034
35	0.887054	8.54206	0.838415	6.84105	0.798953	6.12034
40	0.887054	8.54210	0.838416	6.84106	0.798953	6.12035
45	0.887055	8.54216	0.838416	6.84107	0.798953	6.12035
50	0.887055	8.54220	0.838416	6.84108	0.798953	6.12035
55	0.887055	8.54222	0.838416	6.84108	0.798953	6.12035
60	0.887055	8.54224	0.838416	6.84108	0.798953	6.12035

Together with the overall framework of Secs. 2 and 3, the eigenvalue problem of Eqs. (53*a*)–(53*i*) and the transform-inverse pair of Eqs. (54*a*)–(54*d*) enable the solution of LTNE formulations for roughly the same computational cost as LTE ones. The reason is its capability of collapsing the model into a single transformed problem encompassing the heat transport information in both phases, even when the domains of validity are not the same.

Inevitably, the infinite series of Eqs. (54*b*)–(54*d*) must be truncated at a finite order, M. This way, the accuracy of the fluid and solid temperatures is solely dependent on M. To evaluate the behavior of these quantities with the truncation order, Table 6 shows values of the bulk fluid temperature and the Nusselt number as M varies from 5 to 60. The porous medium has a porosity of 90% and occupies half the channel. The solid phase is assumed to be composed of particles with sizes of 5% of the channel height and thermal conductivity 200 times larger than that of the fluid. Finally, the Péclet number, Pe, is set to 83,405. The convergence is quite remarkable, with a six-significant-digit accuracy being attained with relatively few terms in the truncated series. Given the mostly analytical approach used, the results can be used as a reliable benchmark to verify numerical methods and codes.

Table 7 shows a comparison between the present integral transform analysis and a numerical solution for thermally developed flow in partially porous channels reported in Ref. [126]. Values for the fully developed Nusselt number evaluated at two different Darcy numbers and with the porous medium occupying three different fractions of the channel. The relative deviation between the analytical and numerical solutions is at most 1.22%, further corroborating the adequacy of the proposed integral transform methodology.

Figure 5(*a*) shows a semilogarithmic graph of the Nusselt number (for definition, see Ref. [71]) as a function of the longitudinal coordinate, x. Two values for the porous medium porosity are contemplated. Solid lines and dashed lines represent LTNE and LTE results, respectively. From these results, we can conclude that

Table 7 Comparison of analytical (present) and numerical [126] results for fully developed Nusselt number (LTNE) at different Darcy numbers and with the porous medium occupying different fractions of the channel

		Nu_∞		
Da	y_p	Present work	Ref. [126]	Relative difference
0.001	0.2	1.670	1.673	0.18 %
	0.5	1.369	1.374	0.36 %
	0.8	1.287	1.275	0.94 %
0.0001	0.2	1.645	1.639	0.37 %
	0.5	1.343	1.340	0.22 %
	0.8	1.158	1.144	1.22 %

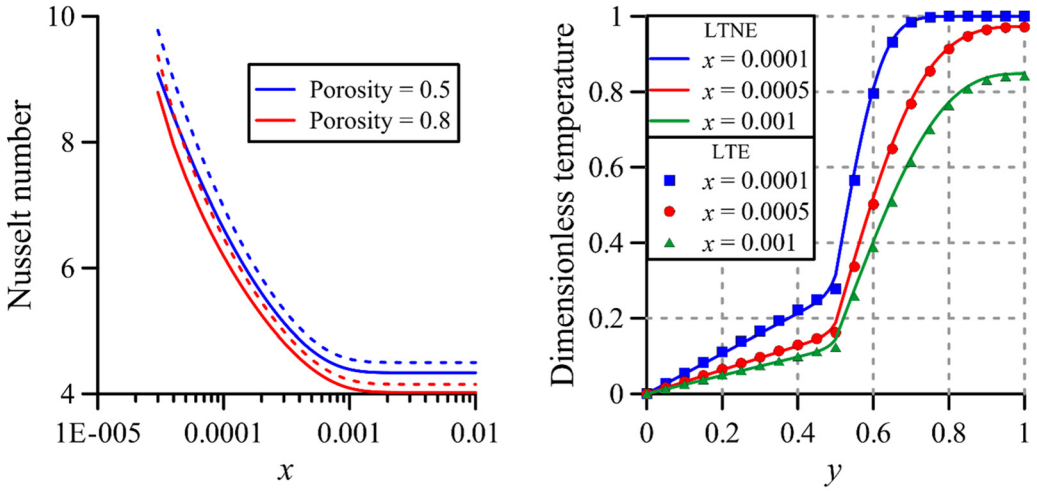

Fig. 5 Nusselt number and fluid temperature profiles. (*a*) Nusselt number—solid lines and dashed lines represent LTNE and LTE results, respectively; (*b*) fluid temperature—solid lines and symbols represent LTNE and LTE results, respectively.

adopting the LTE hypothesis overestimates the heat transfer capability of the apparatus illustrated in Fig. 4, when compared with the more general LTNE formulation, which might be deceiving in applications aiming to improve heat transfer. Since the proposed integral transform framework accomplishes the solution for a very similar computational cost, there is little to no reason to employ the simpler LTE formulation.

Figure 5(*b*) illustrates the fluid temperature profiles for three different longitudinal positions for a porosity equal to 0.8. As can be seen, conductive heat transfer is dominant within most of the porous domain ($y < 0.5$), due to lower fluid velocities there induced by the large hydraulic resistance of the porous material. A comparison between LTNE and LTE fluid temperature profiles shows a smoother transition in the former when compared with the latter at the interface between the free fluid and the porous medium.

Overall, the proposed extension of the solution for the class of problems in Sec. 3.5 is quite effective to cope with problems involving coupled potentials in noncoincident domains.

4.4 Transient Heat Transfer in a Regenerator Channel. The next application deals with convective heat transfer in channels of regenerative heat exchangers. Although conventional models [127] for predicting the thermal performance of such devices are composed of a system of two one-dimensional ODEs and require previous knowledge of the convective heat transfer coefficient between the channel wall and process stream, a modification of this model, that allows for a local description of the temperature field in the flow, can be readily employed, in the form

$$\frac{\partial\theta}{\partial\tau}+\frac{\partial\theta}{\partial\xi}=\mathrm{Pe}^{-2}\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2} \tag{55a}$$

which resembles a transient Gräetz problem with axial diffusion and comprises a simplified version of models previously solved by purely numerical techniques [128].

The model here demonstrated not only assumes a uniform velocity field for simplicity, but also includes the effect of axial diffusion in the fluid, which may become important for shorter channels. The dimensionless variables are given by

$$\theta=\frac{T-T_{\min}}{T_{\max}-T_{\min}} \tag{55b}$$

$$\xi=\frac{x}{L} \tag{55c}$$

$$\eta=\frac{y}{H/2} \tag{55d}$$

$$\tau=\frac{\alpha t}{(H/2)^2} \tag{55e}$$

where the Péclet number is $Pe=\bar{u}(H/2)/\alpha$. The initial and boundary conditions are

$$\theta(\xi,\eta,0)=1 \tag{55f}$$

$$\theta(0,\eta,\tau)=0 \tag{55g}$$

$$\left(\frac{\partial\theta}{\partial\xi}\right)_{\xi=1}=0 \tag{55h}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}=0 \tag{55i}$$

$$C_w\frac{\partial\theta}{\partial\tau}=-\frac{\partial\theta}{\partial\eta},\quad \text{at } \eta=1 \tag{55j}$$

This problem involves the so-called fourth-kind boundary condition at the wall ($\eta=1$), given in terms of the solid-to-fluid heat capacity ratio parameter

$$C_w=\frac{\delta\rho_w c_w}{H\rho c_p} \tag{56}$$

in which δ is the wall thickness, H is the channel spacing, $\rho_w c_w$ is the wall volumetric heat capacity, and ρc_p is the fluid volumetric heat capacity.

The proposed eigenfunction expansion for this problem stems from a couple of one-dimensional eigenvalue problems, given by

$$\frac{d}{d\xi}\left[K(\xi)\frac{dX_i(\xi)}{d\xi}\right]+\gamma_i^2 w(\xi)X_i(\xi)=0 \tag{57a}$$

$$X_i(0)=0 \tag{57b}$$

Table 8 Convergence of the bulk fluid temperature at different axial positions for different dimensionless times in a regenerator channel transient

N	τ	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$
50	0.1	0.0024948	0.0250349	0.238331	0.951330
100	0.1	0.0027347	0.0274098	0.239784	0.949271
200	0.1	0.0028316	0.0283494	0.235972	0.948161
300	0.1	0.0029084	0.0290807	0.234547	0.947705
400	0.1	0.0029474	0.0294434	0.234686	0.947518
500	0.1	0.0029741	0.0296861	0.235086	0.947455
600	0.1	0.0029985	0.0299047	0.235471	0.947537
700	0.1	0.0030166	0.0300635	0.235662	0.947569
N	τ	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$
50	1	0.00120784	0.0121265	0.120681	0.554575
100	1	0.00122452	0.0122926	0.121253	0.554416
200	1	0.00123393	0.0123858	0.121371	0.554417
300	1	0.00123630	0.0124091	0.121366	0.554419
400	1	0.00123739	0.0124198	0.121362	0.554415
500	1	0.00123797	0.0124254	0.121360	0.554417
600	1	0.00123831	0.0124287	0.121359	0.554418
700	1	0.00123853	0.0124309	0.121359	0.554417
N	τ	$\xi=0.001$	$\xi=0.01$	$\xi=0.1$	$\xi=1$
50	10	0.00014970	0.00150364	0.0156287	0.125482
100	10	0.00014973	0.00150400	0.0156324	0.125486
200	10	0.00014974	0.00150409	0.0156333	0.125487
300	10	0.00014975	0.00150412	0.0156335	0.125487
400	10	0.00014975	0.00150413	0.0156336	0.125487
500	10	0.00014975	0.00150413	0.0156337	0.125487
600	10	0.00014975	0.00150413	0.0156337	0.125487
700	10	0.00014975	0.00150413	0.0156337	0.125487

$$\frac{dX_i(1)}{d\xi}=0 \tag{57c}$$

$$\frac{d^2Y_k(\eta)}{d\eta^2}+\mu_k^2Y_k(\eta)=0 \tag{58a}$$

$$\frac{dY_k(0)}{d\eta}=0 \tag{58b}$$

$$\frac{dY_k(1)}{d\eta}-\mu_k^2C_wY_k(1)=0 \tag{58c}$$

for which $K(\xi)=w(\xi)=\exp(-\mathrm{Pe}^2\xi)$. As can be seen, this application illustrates the combined application of a convective eigenvalue problem (in the ξ-direction), Sec. 3.10, and a finite-capacitance boundary condition problem (for the η-direction), Sec. 3.5. These problems admit the following solutions:

$$X_i(\xi)=e^{\mathrm{Pe}^2\xi/2}\sin(\mathrm{Pe}^2\omega_i\xi/2) \tag{59a}$$

$$Y_k(\eta)=\cos(\mu_k\eta) \tag{59b}$$

where the eigenvalues ω_i and μ_k are obtained from the solution of the transcendental equations

$$\sin\left(\mathrm{Pe}^2\omega_i/2\right)+\omega_i\cos\left(\mathrm{Pe}^2\omega_i/2\right)=0 \tag{60a}$$

$$-C_w\mu_k\cos(\mu_k)-\sin(\mu_k)=0 \tag{60b}$$

and $\gamma_i=\left(Pe^2/2\right)\sqrt{1+\omega_i^2}$. The integral transformation pair is then written as

$$\bar{\theta}_{ik}(\tau)=\int_0^1\int_0^1 w(\xi)X_i(\xi)Y_k(\eta)\theta(\xi,\eta,\tau)\mathrm{d}\eta\mathrm{d}\xi+C_wY_k(1)\int_0^1 w(\xi)X_i(\xi)\theta(\xi,1,\tau)\mathrm{d}\xi, \quad \text{transform} \tag{61a}$$

$$\theta(\xi,\eta,\tau)=\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{M_iN_k}X_i(\xi)Y_k(\eta)\bar{\theta}_{ik}(\tau), \quad \text{inverse} \tag{61b}$$

Then, the integral transformation of Eqs. (55*a*) and (55*f*) leads to

$$\frac{d\bar{\theta}_{ik}(\tau)}{d\tau}+\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}E_{ijkl}\bar{\theta}_{jl}(\tau)=0 \tag{62a}$$

$$\bar{\theta}_{ik}(0)=\int_0^1\int_0^1 w(\xi)X_i(\xi)Y_k(\eta)\mathrm{d}\eta\mathrm{d}\xi+C_wY_k(1)\int_0^1 w(\xi)X_i(\xi)\mathrm{d}\xi \tag{62b}$$

in which the matrix coefficients are given by

$$E_{ijkl}=\left(\mu_k^2+\mathrm{Pe}^{-2}\gamma_i^2\right)\delta_{ij}\delta_{kl}-C_w\mathrm{Pe}^{-2}B_{kl}\gamma_i^2\delta_{ij} \tag{62c}$$

$$B_{kl}=\frac{Y_k(1)Y_l(1)}{N_l} \tag{62d}$$

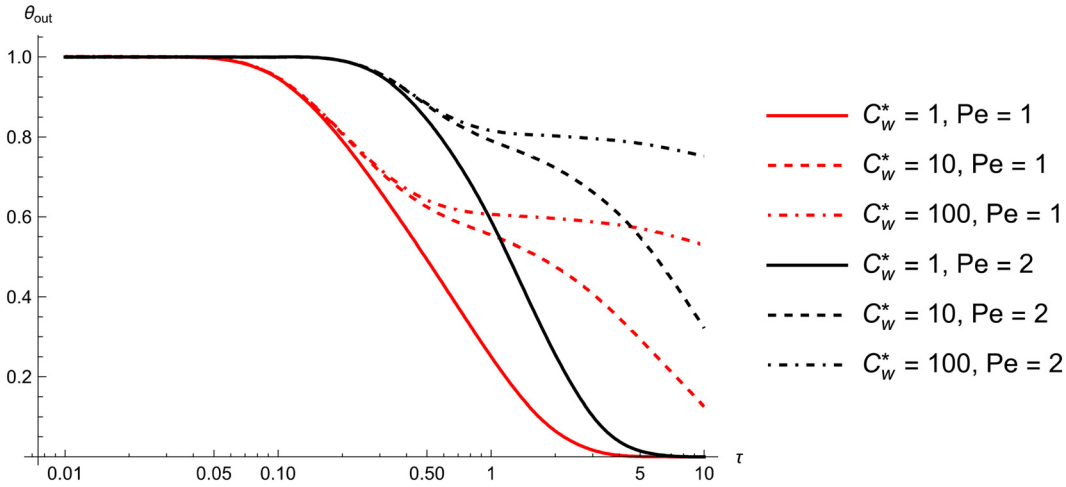

Fig. 6 Outlet bulk temperature evolution for different values of the Péclet number and of the thermal capacity ratio in a regenerator channel transient

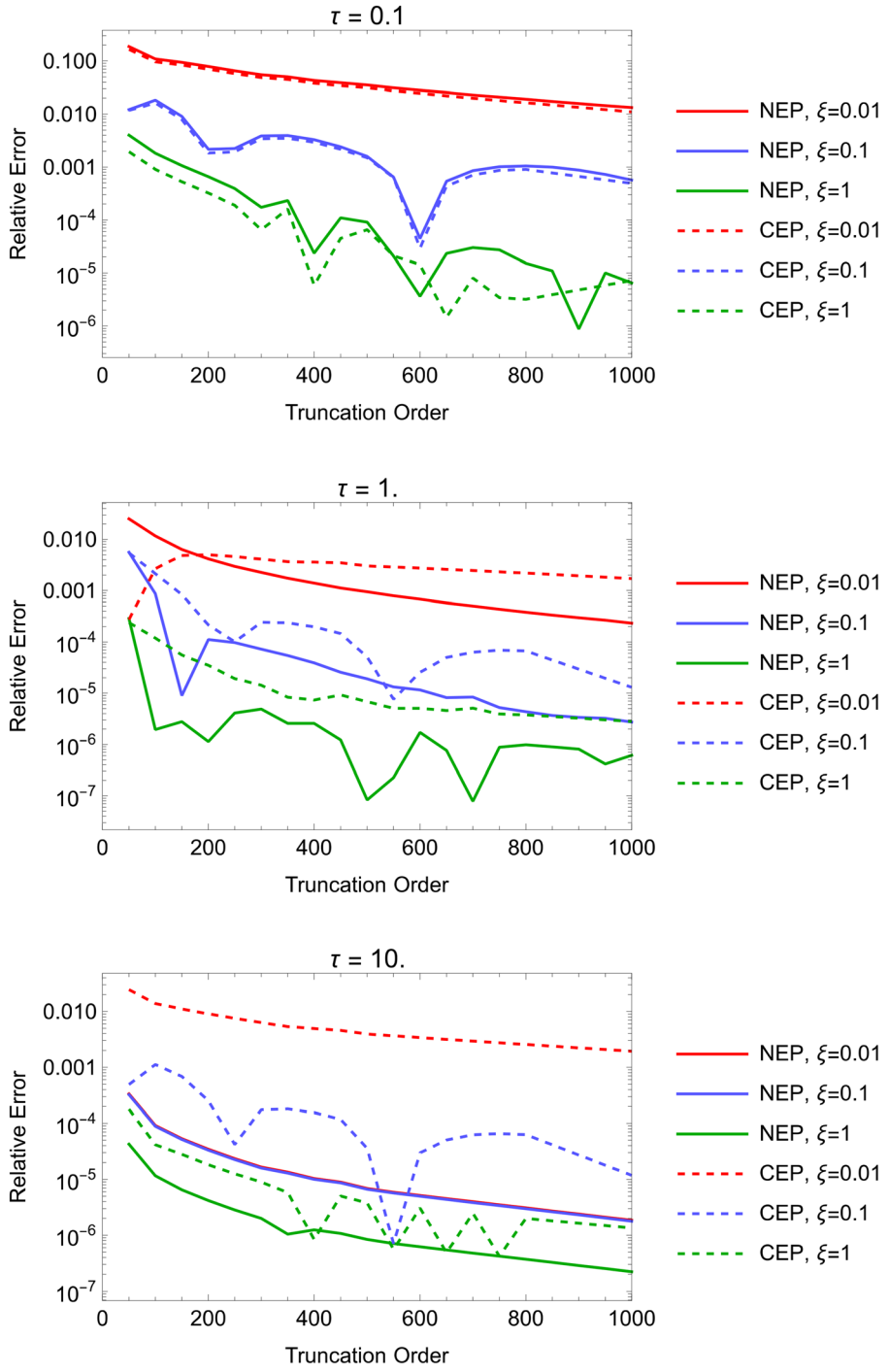

Fig. 7 Evolution of relative error with truncation order for integral transform solutions using a classical eigenvalue problem (CEP) and a nonclassical eigenvalue problem (NEP) for the axial direction

The double series in Eq. (61*b*) is then rewritten as a single summation representation with index $m = m(i,k)$, using the expression $\mu_k^2 + \mathrm{Pe}^{-2}\gamma_i^2$ as a reordering function. This representation allows for the analytical integration of the transformed system, which is computationally implemented using the *Mathematica* MatrixExp function [40].

A brief convergence analysis of the current application is presented in Table 8, which displays the bulk temperature calculated for different axial positions and dimensionless times. N represents the reordered summation truncation order. This temperature is selected for analysis since it is typically used for energy balances in thermal regenerators. As expected, the convergence rate is clearly better for larger dimensionless times, for which there seems to be little influence of the axial position on convergence. On the other hand, for smaller values of τ the convergence rate is improved for downstream positions, a typical behavior in Gräetz-type problems.

Figure 6 displays the outlet bulk temperature calculated for different values of the Péclet number, Pe, and different heat capacity ratios, C_w. For the cases with higher capacity ratios (i.e., the wall has a much greater heat capacity than the fluid), the bulk fluid temperature decreases at a slower rate, as expected. When looking into the effect of the Péclet number values, one clearly notices

that cases with a weaker axial diffusion component (Pe = 2) present a steeper temperature variation when compared to the case with Pe = 1, which is expected, as the axial diffusion component increases the amount of heat that is transported downstream from the hot upstream region. Naturally, this must be examined by contemplating the fact that, as the time scale is diffusion-based, an increase in the Péclet number effectively acts as an augmentation of the flow velocity (advective transport) while maintaining the diffusion velocity.

Finally, for the sake of briefly illustrating the performance improvement achieved by employing a nonconventional eigenvalue problem, the previous results are recalculated using a classical eigenvalue problem (CEP) in the ξ-coordinate, instead of the nonclassical convective eigenvalue problem (NEP) given by Eqs. (57*a*)–(57*c*). The classical problem is obtained from the simple one-dimensional Helmholtz problem $X''(\xi) + \omega^2 X(\xi) = 0$, having the same boundary conditions, as given by Eqs. (57*b*) and (57*c*). Then, to compare the achievable convergence rates, the results are compared by plotting the relative error for each solution for different truncation orders, up to $N = 1000$. This error is evaluated by calculating the difference (in magnitude) for the given truncation order and a reference value, $e_N = |\frac{\theta_N - \theta_{\text{ref}}}{\theta_{\text{ref}}}|$. In all cases, the adopted reference temperature is calculated with 2000 terms in the series using the nonclassical eigenvalue problem. Figure 7 present the calculated relative error results using the two different eigenvalue problems.

As can be seen, the nonclassical eigenvalue problem (NEP) solution clearly shows superior convergence rates, when compared to the classical eigenvalue problem (CEP) one, especially when increasing values of the nondimensional time are considered ($\tau = 1$ and $\tau = 10$). For lower values of the nondimensional time ($\tau = 0.1$), the two solutions roughly display the same convergence rates. As the computational time for each solution is equivalent for the same truncation order, this analysis shows that the nonclassical eigenvalue problem adoption is an enhanced solution path for the given problem.

5 Conclusions

The GITT is reviewed as a hybrid computational–analytical methodology for handling linear or nonlinear partial differential equations in transport phenomena, with emphasis on the employment of nonclassical eigenvalue problems for convergence achievement or enhancement of the associated eigenfunction expansions. The most usual path in application of this methodology has been the adoption of well-known Sturm–Liouville problems as the expansion base and, if required, employing filtering or integral balance schemes to reduce the effects of source terms on convergence rates. These sources include any operators of the original equations and boundary conditions that are not accounted for by the operators of the chosen Sturm–Liouville problem. Therefore, one important path toward convergence enhancement is the consideration of eigenvalue problems that incorporate, at least partially, this additional information that is in fact transferred to the source terms when a simpler eigenfunction expansion base is adopted. This work has thus reviewed the formal GITT methodology and compiled different classes of nonclassical eigenvalue problems that have been employed in association with the integral transform approach throughout the years, leading to successful analytical or hybrid solution implementations. This unification effort is then followed by a section on illustrations of representative test cases that result in excellent convergence rates. This systematic presentation of alternative integral transform solution paths complements in scope the availability of exact analytical solutions [9] and the flexibility of the traditional GITT computational-analytical algorithm [14].

The GITT approach can be said to have been introduced as an approximate analytical solution methodology for nontransformable diffusion problems with time-dependent heat transfer coefficients [10], in this same journal, back in 1974. Almost 50 years have passed by and this approach has reached a mature development stage with a consolidated hybrid numerical–analytical structure. While the different classes of problems of interest to the transport phenomena broad field have already been dealt with, and more recently reaching even other physical areas [115–117,129–130], it remains a challenge to further exploit it in some key areas such as instability analysis, multiphase flows, multiscale problems, simulation under uncertainty, and direct numerical simulation of turbulent flows, to name a few. Again, the choice of expansion base should play a crucial role in the progressive advancement in these directions, which should certainly benefit from the recent breakthroughs, for instance, on single domain reformulation, implicit filtering, vector eigenfunction expansions, and nonlinear eigenvalue problems.

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Nomenclature

d_k = linear dissipation term coefficient, Eq. (1*a*)
f_k = initial condition function, Eq. (1*b*)
$\bar{f}_{ki}$ = transformed initial condition, Eq. (4*b*)
P_k = nonlinear source term, Eq. (1*a*)
$\bar{g}_{ki}$ = transformed source term, Eq. (4*c*)
i = index for eigenfunction/eigenvalue
k = index for number of potentials, Eq. (1*a*)
K_k = diffusion term coefficient, Eq. (1*a*)
n = number of potentials, Eq. (1*a*)
$\mathbf{n}$ = outward-drawn normal to the surface S, problem (1)
N_{ki} = normalization integral (or norm), Eq. (3*d*)
S = domain surface, problem (1)
t = time variable or corresponding spatial variable
T_k = k^{th} potential (dependent variable), problem (1)
$\bar{T}_{ki}$ = transformed k^{th} potential
$\mathbf{T}$ = potentials vector, Eq. (1*d*)
$\mathbf{u}$ = convective term coefficient, Eq. (1*a*)
V = domain region, problem (1)
w_k = transient term coefficient, Eq. (1*a*)
$\mathbf{x}$ = position vector, problem (1)

Greek Symbols

α_k = boundary condition coefficient, Eq. (1*c*)
β_k = boundary condition coefficient, Eq. (1*c*)
λ_{ki} = auxiliary eigenvalues, problem (6)
μ_{ki} = eigenvalues, problem (2)
ϕ_k = nonlinear boundary source term defined in Eq. (1*c*)
ψ_{ki} = eigenfunctions, problem (2)
$\bar{\psi}_{ki}$ = normalized eigenfunctions, Eq. (3*c*)
Ω_{ki} = auxiliary eigenfunctions, problem (6)

References

[1] Cotta, R. M., and Mikhailov, M. D., 2006, "Hybrid Methods and Symbolic Computations," *Handbook of Numerical Heat Transfer*, 2nd ed., W. J. Minkowycz, E. M. Sparrow, and J. Y. Murthy, eds., Wiley, New York, pp. 493–522.
[2] Cotta, R. M., Knupp, D. C., Quaresma, J. N. N, et alet al., 2018, "Analytical Methods in Heat Transfer," *Handbook of Thermal Science and Engineering*, Vol. 1, F. A. Kulacki, ed., Springer, Cham, Switzerland, pp. 61–126.
[3] Cotta, R. M., Knupp, D. C., Lisboa, K. M., Naveira-Cotta, C. P., Quaresma, J. N. N., Zotin, J. L. Z., and Miyagawa, H. K., 2020, "Integral Transform Benchmarks of Diffusion, Convection-Diffusion, and Conjugated Problems in Complex Domains," *50 Years of CFD in Engineering Sciences - A Commemorative*

Volume in Memory of D. Brian Spalding, A. K. Runchal, ed., Springer-Verlag, Singapore, pp. 719–750.
[4] Luikov, A. V., 1968, *Analytical Heat Diffusion Theory*, Academic Press, New York.
[5] Özisik, M. N., 1968, *Boundary Value Problems of Heat Conduction*, Int. Textbooks Co., New York.
[6] Sneddon, IN., 1972, *Use of Integral Transforms*, McGraw-Hill, New York.
[7] Özisik, M. N., 1980, *Heat Conduction*, Wiley, New York.
[8] Luikov, A. V., 1980, *Heat and Mass Transfer*, Mir Publishers, Moscow, Russia.
[9] Mikhailov, M. D., and Özisik, M. N., 1984, *Unified Analysis and Solutions of Heat and Mass Diffusion*, Wiley, New York.
[10] Özisik, M. N., and Murray, R. L., 1974, "On the Solution of Linear Diffusion Problems With Variable Boundary Condition Parameters," ASME J. Heat Transfer, **96**(1), pp. 48–51.
[11] Mikhailov, M. D., 1975, "On the Solution of the Heat Equation With Time Dependent Coefficient," Int. J. Heat Mass Transfer, **18**(2), pp. 344–345.
[12] Cotta, R. M., and Özişik, M. N., 1986, "Laminar Forced Convection in Ducts With Periodic Variation of Inlet Temperature," Int. J. Heat Mass Transfer, **29**(10), pp. 1495–1501.
[13] Cotta, R. M., 1990, "Hybrid Numerical-Analytical Approach to Nonlinear Diffusion Problems," Numer. Heat Transfer, Part B, **17**(2), pp. 217–226.
[14] Cotta, R. M., 1993, *Integral Transforms in Computational Heat and Fluid Flow*, CRC Press, Boca Raton, FL.
[15] Cotta, R. M., 1994, "Benchmark Results in Computational Heat and Fluid Flow: The Integral Transform Method," Int. J. Heat Mass Transfer, **37**(Suppl. 1), pp. 381–393.
[16] Cotta, R. M., and Mikhailov, M. D., 1997, *Heat Conduction: Lumped Analysis, Integral Transforms, Symbolic Computation*, Wiley-Interscience, New York.
[17] Cotta, R. M., 1998, *The Integral Transform Method in Thermal-Fluid Sciences and Engineering*, Begell House, New York.
[18] Cotta, R. M., Knupp, D. C., and Naveira-Cotta, C. P., 2016, *Analytical Heat and Fluid Flow in Microchannels and Microsystems*, Mechanical Engineering Series, Springer-Verlag, Cham, Switzerland.
[19] Cotta, R. M., Naveira-Cotta, C. P., Knupp, D. C., Zotin, J. L. Z., Pontes, P. C., and Almeida, A. P., 2018, "Recent Advances in Computational-Analytical Integral Transforms for Convection-Diffusion Problems," Heat Mass Transfer, **54**(8), pp. 2475–2496.
[20] Cotta, R. M., Lisboa, K. M., Curi, M. F., Balabani, S., Quaresma, J. N. N., Pérez Guerrero, J. S., Macêdo, E. N., and Amorim, N. S., 2019, "A Review of Hybrid Integral Transform Solutions in Fluid Flow Problems With Heat or Mass Transfer and Under Navier-Stokes Equations Formulations," Numer. Heat Transfer, Part B, **76**(2), pp. 60–87.
[21] Mikhailov, M. D., and Özişik, M. N., 1985, "Unified Solutions of Heat Diffusion in a Finite Region Involving a Surface Film of Finite Heat Capacity," Int. J. Heat Mass Transfer, **28**(5), pp. 1039–1045.
[22] Mikhailov, M. D., and Özişik, M. N., 1981, "On General Solution of Heat Conduction in an Anisotropic Medium," Lett. Heat Mass Transfer, **8**(4), pp. 329–335.
[23] Mikhailov, M. D., and Özişik, M. N., 1986, "Unified Finite Integral Transform Analysis," J. Franklin Inst., **321**(6), pp. 299–307.
[24] Vick, B., and Özişik, M. N., 1981, "An Exact Analysis of Low Peclet Number Heat Transfer in Laminar Flow With Axial Conduction," Lett. Heat Mass Transfer, **8**(1), pp. 1–10.
[25] Oliveira, M. C., Ramos, R., and Cotta, R. M., 1995, "On the Eigenvalues Basic to the Analytical Solution of Convective Heat Transfer With Axial Diffusion Effects," Commun. Numer. Methods Eng., **11**(4), pp. 287–296.
[26] Aperecido, J. B., Cotta, R. M., and Özişik, M. N., 1989, "Analytical Solutions to Two-Dimensional Diffusion Type Problems in Irregular Geometries," J. Franklin Inst., **326**(3), pp. 421–434.
[27] Sphaier, L. A., and Cotta, R. M., 2000, "Integral Transform Analysis of Multidimensional Eigenvalue Problems Within Irregular Domains," Numer. Heat Transfer, Part B, **38**(2), pp. 157–175.
[28] Diniz, A. J., Aparecido, J. B., and Cotta, R. M., 1990, "Heat Conduction With Ablation in a Finite Slab," Int. J. Heat Technol., **8**, pp. 30–43.
[29] Sias, D. F., Ruperti, N. J., Jr., and Cotta, R. M., 2009, "Enhanced Convergence of Integral Transform Solution of Ablation Problems," High Temp., High Pressures, **38**(2), pp. 81–96.
[30] Pérez Guerrero, J. S., and Cotta, R. M., 1992, "Integral Transform Method for Navier-Stokes Equations in Stream Function-Only Formulation," Int. J. Numer. Methods Fluids, **15**(4), pp. 399–409.
[31] Figueira da Silva, E., and Cotta, R. M., 1996, "Benchmark Results for Internal Forced Convection Through Integral Transformation," Int. Commun. Heat Mass Transfer, **23**(7), pp. 1019–1029.
[32] Knupp, D. C., Cotta, R. M., and Naveira-Cotta, C. P., 2015, "Fluid Flow and Conjugated Heat Transfer in Arbitrarily Shaped Channels Via Single Domain Formulation and Integral Transforms," Int. J. Heat Mass Transfer, **82**, pp. 479–489.
[33] Almeida, A. P., Naveira-Cotta, C. P., and Cotta, R. M., 2020, "Transient Multidimensional Heat Conduction in Heterogeneous Media: Integral Transform With Single Domain Formulation," Int. Comm. Heat Mass Transfer, **117**, p. 104792.
[34] Cotta, R. M., Naveira-Cotta, C. P., and Knupp, D. C., 2017, "Convective Eigenvalue Problems for Convergence Enhancement of Eigenfunction Expansions in Convection-Diffusion Problems," ASME J. Therm. Sci. Eng. Appl., **10**(2), p. 021009.
[35] Knupp, D. C., Cotta, R. M., and Naveira-Cotta, C. P., 2020, "Conjugated Heat Transfer Analysis Via Integral Transforms and Eigenvalue Problems," J. Eng. Phys. Thermophys., **93**(1), pp. 65–77.
[36] Lisboa, K. M., and Cotta, R. M., 2018, "Hybrid Integral Transforms for Flow Development in Ducts Partially Filled With Porous Media," Proc. R. Soc. A, **474**, pp. 1–20.
[37] Lisboa, K. M., Su, J., and Cotta, R. M., 2019, "Vector Eigenfunction Expansion in the Integral Transform Solution of Transient Natural Convection," Int. J. Numer. Methods Heat Fluid Flow, **29**(8), pp. 2684–2708.
[38] Cotta, R. M., Naveira-Cotta, C. P., and Knupp, D. C., 2016, "Nonlinear Eigenvalue Problem in the Integral Transforms Solution of Convection-Diffusion With Nonlinear Boundary Conditions," Int. J. Numer. Methods Heat Fluid Flow, **26**(3/4), pp. 767–789.
[39] Pontes, P. C., Almeida, A. P., Cotta, R. M., and Naveira-Cotta, C. P., 2018, "Analysis of Mass Transfer in Hollow-Fiber Membrane Separator Via Nonlinear Eigenfunction Expansions," Multiphase Sci. Technol., **30**(2–3), pp. 165–186.
[40] Wolfram, S., 2021, *Mathematica Version 12.3*, Wolfram Research, Champaign, IL.
[41] Mikhailov, M. D., and Cotta, R. M., 1994, "Integral Transform Method for Eigenvalue Problems," Commun. Numer. Methods Eng., **10**(10), pp. 827–835.
[42] Naveira-Cotta, C. P., Cotta, R. M., Orlande, H. R. B., and Fudym, O., 2009, "Eigenfunction Expansions for Transient Diffusion in Heterogeneous Media," Int. J. Heat Mass Transfer, **52**(21–22), pp. 5029–5039.
[43] Almeida, A. R., and Cotta, R. M., 1996, "A Comparison of Convergence Acceleration Schemes for Eigenfunction Expansions of Partial Differential Equations," Int. J. Numer. Methods Heat Fluid Flow, **6**(6), pp. 85–97.
[44] Mikhailov, M. D., and Cotta, R. M., 2001, "Convergence Acceleration of Integral Transform Solutions," Hybrid Meth. Eng., **3**(1), pp. 89–94.
[45] Cotta, R. M., Naveira-Cotta, C. P., and Knupp, D. C., 2016, "Enhanced Eigenfunction Expansions in Convection-Diffusion Problems With Multiscale Space Variable Coefficients," Numer. Heat Transfer, Part A, **70**(5), pp. 492–512.
[46] Cotta, R. M., Knupp, D. C., Naveira-Cotta, C. P., Sphaier, L. A., and Quaresma, J. N. N., 2014, "The Unified Integral Transforms (UNIT) Algorithm With Total and Partial Transformation," Comput. Therm. Sci., **6**(6), pp. 507–524.
[47] Carvalho, I. S., Cotta, R. M., Naveira-Cotta, C. P., and Tiwari, M. K., 2021, "Hybrid Integral Transform Analysis of Supercooled Droplets Solidification," Proc. R. Soc. A, **477**, pp. 1–25.
[48] Hirata, S. C., Goyeau, B., Gobin, D., Carr, M., and Cotta, R. M., 2007, "Linear Stability of Natural Convection in Superposed Fluid and Porous Layers: Influence of the Interfacial Modelling," Int. J. Heat Mass Transfer, **50**(7–8), pp. 1356–1367.
[49] Hirata, S. C., Goyeau, B., Gobin, D., Chandesris, M., and Jamet, D., 2009, "Stability of Natural Convection in Superposed Fluid and Porous Layers: Equivalence of the One- and Two-Domain Approaches," Int. J. Heat Mass Transfer, **52**(1–2), pp. 533–536.
[50] Knupp, D. C., Naveira-Cotta, C. P., and Cotta, R. M., 2012, "Theoretical Analysis of Conjugated Heat Transfer With a Single Domain Formulation and Integral Transforms," Int. Commun. Heat Mass Transfer, **39**(3), pp. 355–362.
[51] Knupp, D. C., Naveira-Cotta, C. P., and Cotta, R. M., 2013, "Conjugated Convection-Conduction Analysis in Microchannels With Axial Diffusion Effects and a Single Domain Formulation," ASME J. Heat Transfer, **135**(9), p. 091401.
[52] Knupp, D. C., Cotta, R. M., Naveira-Cotta, C. P., and Kakaç, S., 2015, "Transient Conjugated Heat Transfer in Microchannels: Integral Transforms With Single Domain Formulation," Int. J. Therm. Sci., **88**, pp. 248–257.
[53] Lisboa, K. M., and Cotta, R. M., 2018, "On the Mass Transport in Membraneless Flow Batteries of Flow-by Configuration," Int. J. Heat Mass Transfer, **122**, pp. 954–966.
[54] Lisboa, K. M., Su, J., and Cotta, R. M., 2018, "Single Domain Integral Transforms Analysis of Natural Convection in Cavities Partially Filled With Heat Generating Porous Medium," Numer. Heat Transfer, Part A, **74**(3), pp. 1068–1086.
[55] Knupp, D. C., Mascouto, F. S., Abreu, L. A. S., Naveira-Cotta, C. P., and Cotta, R. M., 2018, "Conjugated Heat Transfer in Circular Microchannels With Slip Flow and Axial Diffusion Effects," Int. Commun. Heat Mass Transfer, **91**, pp. 225–233.
[56] Silva, G. R., Knupp, D. C., Naveira-Cotta, C. P., Cotta, R. M., and Silva Neto, A. J., 2020, "Estimation of Slip Flow Parameters in Microscale Conjugated Heat Transfer Problem," J. Braz. Soc. Mech. Sci. Eng., **42**(5), p. 263.
[57] Moon, P., and Spencer, D. E., 1961, *Field Theory for Engineers*, Van Nostrand, Princeton, NJ.
[58] Sphaier, L. A., and Cotta, R. M., 2002, "Analytical and Hybrid Solutions of Diffusion Problems Within Arbitrarily Shaped Regions Via Integral Transforms," Comput. Mech., **29**(3), pp. 265–276.
[59] Sphaier, L. A., and Cotta, R. M., 2002, "Integral Transform Solutions of Diffusion and Eigenvalue Problems Within Arbitrarily Shaped Domains," Proceedings of the 12th International Heat Transfer Conference (IHTC 12), Grenoble, France, Aug. 18–23, Paper No. 0908.
[60] Maia, C. R. M., Aparecido, J. B., and Milanez, L. F., 2006, "Heat Transfer in Laminar Flow of Non-Newtonian Fluids in Ducts of Elliptical Section," Int. J. Therm. Sci., **45**(11), pp. 1066–1072.

[61] Pinheiro, I. F., Sphaier, L. A., and Knupp, D. C., 2019, "Integral Transform Solution of Eigenvalue Problems Within Irregular Geometries: Comparative Analysis of Different Methodologies," Numer. Heat Transfer, Part B, **76**(6), pp. 329–350.
[62] Aparecido, J. B., and Cotta, R. M., 1992, "Laminar Thermally Developing Flow Inside Right Triangular Ducts," Appl. Sci. Res., **49**(4), pp. 355–368.
[63] Barbuto, F. A. A., and Cotta, R. M., 1997, "Integral Transformation of Elliptic Problems Within Irregular Domains: Fully Developed Channel Flow," Int. J. Numer. Methods Heat Fluid Flow, **7**(8), pp. 778–793.
[64] Cotta, R. M., and Ramos, R., 1998, "Integral Transforms in the Two-Dimensional Nonlinear Formulation of Longitudinal Fins With Variable Profile," Int. J. Numer. Methods Heat Fluid Flow, **8**(1), pp. 27–42.
[65] Pérez Guerrero, J. S., Quaresma, J. N. N., and Cotta, R. M., 2000, "Simulation of Laminar Flow Inside Ducts of Irregular Geometry Using Integral Transforms," Comput. Mech., **25**(4), pp. 413–420.
[66] Monteiro, E. R., Macêdo, E. N., Quaresma, J. N. N., and Cotta, R. M., 2010, "Laminar Flow and Convective Heat Transfer of Non-Newtonian Fluids in Doubly Connected Ducts," Int. J. Heat Mass Transfer, **53**(11–12), pp. 2434–2448.
[67] Miyagawa, H. K., Quaresma, J. N. N., Lisboa, K. M., and Cotta, R. M., 2020, "Integral Transform Analysis of Convective Heat Transfer Within Wavy Walls Channels," Numer. Heat Transfer, Part A, **77**(5), pp. 460–481.
[68] Mikhailov, M. D., and Cotta, R. M., 1990, "Unified Integral Transform Method," J. Braz. Soc. Mech. Sci., **12**(3), pp. 301–310.
[69] Mikhailov, M. D., Özisik, M. N., and Shishedjiev, B. K., 1982, "Diffusion in Heterogeneous Media," ASME J. Heat Transfer, **104**(4), pp. 781–787.
[70] Cotta, R. M., Lisboa, K. M., and Zotin, J. L. Z., 2020, "Integral Transforms for Flow and Transport in Continuum and Discrete Models of Fractured Heterogeneous Porous Media," Adv. Water Resour., **142**, p. 103621.
[71] Lisboa, K. M., Zotin, J. L. Z., and Cotta, R. M., 2021, "Hybrid Solutions for Thermally Developing Flows in Channels Partially Filled With Porous Media," Numer. Heat Transfer, Part B, **79**(4), pp. 189–215.
[72] Mikhailov, M. D., 1973, "General Solutions of the Coupled Diffusion Equations," Int. J. Eng. Sci., **11**(2), pp. 235–241.
[73] Alves, L. S. B., Cotta, R. M., and Pontes, J., 2002, "Stability Analysis of Natural Convection in Porous Cavities Through Integral Transforms," Int. J. Heat Mass Transfer, **45**(6), pp. 1185–1195.
[74] Sphaier, L. A., and Barletta, A., 2014, "Unstable Mixed Convection in a Heated Horizontal Porous Channel," Int. J. Therm. Sci., **78**, pp. 77–89.
[75] Sphaier, L. A., Barletta, A., and Celli, M., 2015, "Unstable Mixed Convection in a Heated Inclined Porous Channel," J. Fluid Mech., **778**, pp. 428–450.
[76] Mikhailov, M. D., 1973, "General Solutions of the Diffusion Equations Coupled at Boundary Conditions," Int. J. Heat Mass Transfer, **16**(12), pp. 2155–2164.
[77] Mikhailov, M. D., and Shishedjiev, B. K., 1975, "Temperature and Moisture Distributions During Contact Drying of a Moist Porous Sheet," Int. J. Heat Mass Transfer, **18**(1), pp. 15–24.
[78] Mikhailov, M. D., and Shishedjiev, B. K., 1976, "Coupled at Boundary Mass or Heat Transfer in Entrance Concurrent Flow," Int. J. Heat Mass Transfer, **19**(5), pp. 553–557.
[79] Lobo, P. D., Mikhailov, M. D., and Özisik, M. N., 1987, "On the Complex Eigenvalues of Luikov System of Equations," Drying Technol., **5**(2), pp. 273–286.
[80] Ribeiro, J. W., Cotta, R. M., and Mikhailov, M. D., 1993, "Integral Transform Solution of Luikov's Equations for Heat and Mass Transfer in Capillary Porous Media," Int. J. Heat Mass Transfer, **36**(18), pp. 4467–4475.
[81] Ribeiro, J. W., and Cotta, R. M., 1995, "On the Solution of Nonlinear Drying Problems in Capillary Porous Media Through Integral Transformation of Luikov Equations," Int. J. Numer. Methods Eng., **38**(6), pp. 1001–1020.
[82] Conceição, R. S. G., Macêdo, E. N., Pereira, L. B. D., and Quaresma, J. N. N., 2013, "Hybrid Integral Transform Solution for the Analysis of Drying in Spherical Capillary-Porous Solids Based on Luikov Equations With Pressure Gradient," Int. J. Therm. Sci., **71**, pp. 216–236.
[83] Guigon, S. M., Dantas, L. B., Scofano Neto, F., and Cotta, R. M., 1999, "Exact Solution of Luikov's Equations for Drying in Capillary Porous Media," Hybrid Methods Eng., **1**(4), pp. 365–387.
[84] Cotta, R. M., and Özisik, M. N., 1986, "Thermally Developing Concurrent-Flow Circular Double-Pipe Heat Exchanger Analysis," Proceedings of the Eighth International Heat Transfer Conference, San Francisco, CA, Aug. 17–22, pp. 2805–2810.
[85] Soares, F. A., Guedes, R. O. C., and Scofano Neto, F., 2018, "Analysis of the Influence of Pellet-to-Cladding Gap on the Transient Heat Transfer in Nuclear Fuel Rods Via the Integral Transform Technique," J. Braz. Soc. Mech. Sci. Eng., **40**, p. 308.
[86] Matt, C. F. T., 2015, "Transient Response of General One-Dimensional Distributed System Through Eigenfunction Expansion With an Implicit Filter Scheme," Appl. Math. Modell., **39**(9), pp. 2470–2488.
[87] Hirata, S. C., Couto, P., Lara, L. G., Cotta, R. M., 2009, Modelling and Hybrid Simulation of Slow Discharge Process of Adsorbed Methane Tanks, Int. J. Thermal Sciences, **48**, pp. 1176–1183.
[88] Silva, E. M., Quaresma, J. N. N., Macêdo, E. N., and Cotta, R. M., 2021, "Integral Transforms for Three-Dimensional Pumping in Confined, Leaky, and Unconfined Aquifers," J. Hydrol. Hydromech., **69**(3), pp. 319–331.
[89] Mikhailov, M. D., 1977, "General Solution of Diffusion Processes in Solid-Liquid Extraction," Int. J. Heat Mass Transfer, **20**(12), pp. 1409–1415.
[90] Scofano Neto, F., and Cotta, R. M., 1992, "Lumped-Differential Analysis of Concurrent Flow Double-Pipe Heat Exchanger," Can. J. Chem. Eng., **70**(3), pp. 592–595.
[91] Scofano Neto, F., and Cotta, R. M., 1992, "Counterflow Double-Pipe Heat Exchangers Analysis Through a Mixed Lumped-Differential Formulation," Int. J. Heat Mass Transfer, **35**(7), pp. 1723–1731.
[92] Scofano Neto, F., and Cotta, R. M., 1993, "Improved Hybrid Lumped-Differential Formulation for Double-Pipe Heat Exchangers Analysis," ASME J. Heat Transfer, **115**(4), pp. 921–927.
[93] Isaza, P. A., and Bussmann, M., 2021, "An Analytical Solution for Moving Bed Heat Exchangers Via Integral Transform Methods," Heat Transfer Eng., **42**(3–4), pp. 215–222.
[94] Makaveev, B. R., and Mikhailov, M. D., 2001, "An Eigenvalue Problem With Nonlinear Dependence of the Eigenvalue," Hybrid Methods Eng., **3**(4), pp. 409–413.
[95] Bayazitoglu, Y., and Ozisik, M. N., 1980, "On the Solution of Gräetz Type Problems With Axial Conduction," Int. J. Heat Mass Transfer, **23**(10), pp. 1399–1402.
[96] Gondim, R. R., Macêdo, E. N., and Cotta, R. M., 2007, "Hybrid Solution for Transient Internal Convection With Axial Diffusion: Integral Transforms With Local Instantaneous Filtering," Int. J. Numer. Methods Heat Fluid Flow, **17**(4), pp. 405–417.
[97] Sphaier, L. A., Braga, N. R., Jr., and Chalhub, D., 2021, "Analytical Solutions for Extended Gräetz Problem in Infinite Domains Via Integral Transforms," Int. J. Therm. Sci., **170**, p. 107093.
[98] Knupp, D. C., Cotta, R. M., and Naveira-Cotta, C. P., 2013, "Heat Transfer in Microchannels With Upstream-Downstream Regions Coupling and Wall Conjugation Effects," Numer. Heat Transfer, Part B, **64**(5), pp. 365–387.
[99] Cotta, R. M., Serfaty, R., and Guedes, R. O. C., 1990, "Integral Transform Solution of a Class of Transient Convection-Diffusion Problems," Proceedings of the International Conference on Advanced Computational Methods in Heat Transfer, Southampton, UK, July 17–20, pp. 239–250.
[100] Serfaty, R., and Cotta, R. M., 1992, "Hybrid Analysis of Transient Nonlinear Convection-Diffusion Problems," Int. J. Numer. Methods Heat Fluid Flow, **2**(1), pp. 55–62.
[101] Leiroz, A. J. K., and Cotta, R. M., 1993, "On the Solution on Nonlinear Elliptic Convection-Diffusion Problems Through the Integral Transform Method," Numer. Heat Transfer, Part B, **23**(4), pp. 401–411.
[102] Alves, L. S. B., Cotta, R. M., and Mikhailov, M. D., 2001, "Covalidation of Hybrid Integral Transforms and Method of Lines in Nonlinear Convection-Diffusion With Mathematica," J. Braz. Soc. Mech. Sci., **23**(3), pp. 303–319.
[103] Pérez Guerrero, J. S., Pimentel, L. C. G., Skaggs, T. H., and van Genuchten, M. T., 2009, "Analytical Solution of the Advection-Diffusion Transport Equation Using a Change-of-Variable and Integral Transform Technique," Int. J. Heat Mass Transfer, **52**(13–14), pp. 3297–3304.
[104] Knupp, D. C., Lisboa, K. M., and Cotta, R. M., 2020, "Convective-Diffusive Eigenvalue Problem Bases for Integral Transform Solutions in Transport Phenomena," Proceedings of the 18th Brazilian Congress of Thermal Sciences and Engineering, Porto Alegre, RS, Brazil, Nov. 16–20, Paper No. ENC-2020-0292.
[105] Machado, H. A., and Cotta, R. M., 1995, "Integral Transform Method for Boundary Layer Equations in Simultaneous Heat and Fluid Flow Problems," Int. J. Numer. Methods Heat Fluid Flow, **5**(3), pp. 225–237.
[106] Bolivar, M. A. H., Lage, P. L. C., and Cotta, R. M., 1998, "Integral Transform Solution of the Laminar Thermal Boundary Layer Problem for Flow Past Two-Dimensional and Axisymmetric Bodies," Numer. Heat Transfer, Part A, **33**(7), pp. 779–797.
[107] Cotta, R. M., and Pimentel, L. C. G., 1998, "Developing Turbulent Duct Flow: Hybrid Solution Via Integral Transforms and Algebraic Models," Int. J. Numer. Methods Heat Fluid Flow, **8**(1), pp. 10–26.
[108] Lima, G. G. C., Santos, C. A. C., Haag, A., and Cotta, R. M., 2007, "Integral Transform Solution of Internal Flow Problems Based on Navier-Stokes Equations and Primitive Variables Formulation," Int. J. Numer. Methods Eng., **69**(3), pp. 544–561.
[109] Pérez Guerrero, J. S., and Cotta, R. M., 1995, "Integral Transform Solution of Developing Laminar Duct Flow in Navier-Stokes Formulation," Int. J. Numer. Methods Fluids, **20**(11), pp. 1203–1213.
[110] Pérez Guerrero, J. S., and Cotta, R. M., 1996, "Benchmark Integral Transform Results for Flow Over a Backward-Facing Step," Comput. Fluids, **25**(5), pp. 527–540.
[111] Quaresma, J. N. N., and Cotta, R. M., 1997, "Integral Transform Method for the Navier-Stokes Equations in Steady Three-Dimensional Flow," Proceedings of the Tenth ISTP - International Symposium on Transport Phenomena, Kyoto, Japan, Nov., pp. 281–287.
[112] Pereira, L. M., Pérez Guerrero, J. S., and Cotta, R. M., 1998, "Integral Transformation of the Navier-Stokes Equations in Cylindrical Geometry," Comput. Mech., **21**(1), pp. 60–70.
[113] Pereira, L. M., Pérez Guerrero, J. S., Brazão, N., and Cotta, R. M., 2002, "Compressible Flow and Heat Transfer in Ultracentrifuges: Hybrid Analysis Via Integral Transforms," Int. J. Heat Mass Transfer, **45**(1), pp. 99–112.
[114] Meirovitch, L., 1967, *Analytical Methods in Vibrations*, MacMillan, New York.
[115] Matt, C. F. T., 2009, "On the Application of Generalized Integral Transform Technique to Wind-Induced Vibrations on Overhead Conductors," Int. J. Numer. Methods Eng., **78**(8), pp. 901–930.

[116] Gu, J., An, C., Levi, C., and Su, J., 2012, "Prediction of Vortex-Induced Vibration of Long Flexible Cylinders Modeled by a Coupled Nonlinear Oscillator: Integral Transform Solution," J. Hydrodyn., **24**(6), pp. 888–898.
[117] Matt, C. F. T., 2013, "Simulation of the Transverse Vibrations of a Cantilever Beam With an Eccentric Tip Mass in the Axial Direction Using Integral Transforms," Appl. Math. Modell., **37**(22), pp. 9338–9354.
[118] dos Reis, E. V. M., Sphaier, L. A., Nunes, L. C. S., and Alves, L. S. B., 2018, "Dynamic Response of Free Span Pipelines Via Linear and Nonlinear Stability Analyses," Ocean Eng., **163**, pp. 533–543.
[119] He, Y., An, C., and Su, J., 2020, "Generalized Integral Transform Solution for Free Vibration of Orthotropic Rectangular Plates With Free Edges," J. Braz. Soc. Mech. Sci. Eng., **42**, p. 183.
[120] Cordeiro, C. E. Z., Stutz, L. T., Knupp, D. C., and Matt, C. F. T., 2022, "Generalized Integral Transform and Hamiltonian Monte Carlo for Bayesian Structural Damage Identification," Appl. Math. Modell., **104**, pp. 243–258.
[121] Cotta, R. M., Naveira-Cotta, C. P., and Knupp, D. C., 2015, "Nonlinear Eigenfunction Expansions for the Solution of Nonlinear Diffusion Problems," Proceedings of the First Thermal and Fluid Engineering Summer Conference, ASTFE Digital Library, New York, Aug. 9–12, Paper No. TFESC-13652.
[122] Cotta, R. M., Lisboa, K. M., Pontes, P. C., Naveira-Cotta, C. P., Quaresma, J. N. N., and Macêdo, E. N., 2019, "Hybrid Integral Transforms in Nonlinear Heat and Mass Transfer," Proceedings of the Sixth International Conference on Thermophysical and Mechanical Properties of Advanced Materials (THERMAM), Çeşme, İzmir, Turkey, Sept. 22–24, pp. 180–200.
[123] Lamien, B., and Orlande, H. R. B., 2013, "Simultaneous Estimation of Thermal Conductivity and Volumetric Heat Capacity of Viscous Liquids With the Line Heat Source Probe Via Bayesian Inference," High Temp., High Pressures, **42**(3), pp. 151–174.
[124] Lamien, B., and Orlande, H. R. B., 2013, "Approximation Error Model to Account for Convective Effects in Liquids Characterized by the Line Heat Source Probe," Proceedings of the Fourth Inverse Problems, Design and Optimization Symposium (IPDO-2013), Albi, France, June 26–28, pp. 1–10.
[125] Mahmoudi, Y., and Karimi, N., 2014, "Numerical Investigation of Heat Transfer Enhancement in a Pipe Partially Filled With a Porous Material Under Local Thermal Non-Equilibrium Condition," Int. J. Heat Mass Transfer, **68**, pp. 161–173.
[126] Poulikakos, D., and Kazmierczak, M., 1987, "Forced Convection in a Duct Partially Filled With a Porous Material," ASME J. Heat Transfer, **109**(3), pp. 653–662.
[127] Shah, R. K., and Sekulic, D. P., 2002, *Fundamentals of Heat Exchanger Design*, Wiley, New York.
[128] Santos, S. M. D., and Sphaier, L. A., 2020, "Transient Formulation for Evaluating Convective Coefficients in Regenerative Exchangers With Hygroscopic Channels," Int. Commun. Heat Mass Transfer, **116**, p. 104691.
[129] Li, F., An, C., Duan, M., and Su, J., 2020, "Combined Damping Model for Dynamics and Stability of a Pipe Conveying Two-Phase Flow," Ocean Eng., **195**, p. 106683.
[130] He, Y., An, C., and Su, J., 2020, "Generalized Integral Transform Solution for Free Vibration of Orthotropic Rectangular Plates With Free Edges," J. Braz. Soc. Mech. Sci. Eng., **42**(4), p. 183.