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Application of the Boundary-Value Technique to Singular Perturbation Problems at High Reynolds Numbers

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The boundary-value technique, advanced by Roberts for the solution of singular pertubation problems of ordinary differential equations where the small parameter multiples the highest derivative, is extended to the solution of the Navier–Stokes equation at high Reynolds numbers. Three standard flows—uniform flow past a plate, flow with a linearly adverse external velocity, and shear flow past a flat plate—have been chosen as test problems with a view to evaluating some of the features of the boundary-value technique, particularly in comparison with coefficient matching techniques as examplified by the method of matcher asymptotic expansions. © 1987 Academic Press, Inc.

1. INTRODUCTION

Singular perturbation problems pervade fluid mechanics in a variety of forms and contexts. In flow Reynolds number flow the singularity arises due to the fact that inertial effects are small but not negligible far away from the body, and in boundary layer flows at high Reynolds numbers, the perturbation is singular due to the fact that viscous effects can be neglected everywhere except within a thin layer next to the body. A number of techniques have been developed and applied to singular perturbation problems in general and in fluid mechanics in particular, see [1, 2, 3]. The

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common feature of those techniques is that the flow is divided into two regions, called inner and outer, in each of which an asymptotic expansion is determined, and these two expansions are combined to obtain a uniformly valid solution, the precess of combining the pair of expansions being performed through matching of their corresponding coefficients. Two aspects stand out in those so-called coefficient matching techniques (CMT): selecting the appropriate asymptotic expansions for the specific problems under consideration and matching the expansions once they have been determined. The choice of the appropriate asympotic expansions is far from straightforward as it involves a good measure of experience, insight, and intuition. Further, the matching of the expansions through their coefficients involves a numbers of conceptual difficulties.

With a view to avoiding these difficulties altogether, Roberts [4] has presented a method, which he denotes the boundary value technique (BVT), for the solution of singular perturbation problems of ordinary differential equations, where the small perturbation parameter multiplies the highest derivative. In the boundary-value technique, the interval over which the problem is formulated is divided into two intervals, inner and outer, in each of which a solution is obtained by analytical, numerical, or asymptotic methods. The outer solution, corresponding to the outer interval, constitutes the terminal condition for the inner solution which is valid in the inner region. The inner problem is then solved iteratively for various values of the terminal point of the inner interval until the successive iterations of the solutions are close to each other within the desired degree of accuracy. Herein lies the principal difference between the coefficient matching techniques on the one hand the boundary-value technique on the other. The adjoining of the outer and the inner solutions in the BVT is carried out at a *point* in the domain of the problem, this point being found iteratively, while in the CMT the inner and the outer expansions are matched *asymptotically*.

In addition to the above-mentioned advantages of avoiding altogether the far from simple problem of the choice of asymptotic expansion and the intricate question of matching the inner and the outer expansions, in general there are a number of significant computational differences associated with the BVT in relation to the CMT. In the first place, once the original problem is divided into its inner and outer parts, one is faced with the task of solving this pair of problems in the BVT and a sequence of problem pairs (zero-order, first-order, etc.) in the CMT. Generally, less computational effort would be involved in the former than in the latter situation, although it should not be forgotten that the BVT involves an iterative scheme and that the problem pair arising in the BVT is normally more complex than the problem pairs corresponding to the various orders in the CMT. A very important advantage that is obtained with the BVT compared with the CMT arises when dealing with problems involving infinite intervals. In the BVT, one has to solve the outer problem as an initial-value problem and the inner problem as a multi-point boundary-value problem over a *finite* interval. This is in contrast to the situation that arises in the CMT, where the outer problems are initial-value but the inner problems are multi-point boundary-value over an *infinite* interval. Clearly, it is simpler mathematically and computationally to deal with multi-point boundary-value problems in finite rather than in infinite domains.

The objective of this paper is to extend the application of the BVT to some singular pertubations of the Navier–Stokes equations. In order to evaluate the results, we have selected flows for which solutions have been obtained by classical boundary layer theory and CMT. In Section 2, we give a general description of the BVT as applied to singular perturbation problems of partial differential equations. This is done since Roberts [4] presented the technique in the context of ordinary differential equations. Two boundary layer-type flows are determined in Section 3, using the BVT, and the results are compared with the established solutions of Blassius and Howarth [5] for uniform flow past a flat plate and flow with a linearly adverse external velocity, respectively. In Section 4, a hybrid method, involving the BVT and the technique of asymptotic expansions, is utilized to determine shear flow past a flat plate. This problem is also solved by use of the BVT alone and results are compared with the solutions of Murray [6]. This is done with a view to demonstrating the potential of the BVT in so far as its couplability with other methods and suggestions are presented with respect to the extension of the BVT to more complex flows and to deal with other types of singularities in fluid mechanics.

2. Description of the Boundary-Value Method for Singular Perturbation Problems

Consider a partial differential equation of the form

$$f(\varepsilon, x, y, \ldots, \mu, \mu_x, \mu_y, \ldots, \mu_{xx}, \mu_{xy}, \ldots) = 0, \tag{1}$$

where the small parameter ε multiplies a term containing one of the highest derivatives, in x say. We seek a solution to Eq. (1) satisfying m boundary conditions in x, the validity of this solution being for small values of the parameter ε.

Letting $\varepsilon \to 0$ results in a lowering of the order of the equation, and we obtain the so-called outer equation

$$\begin{aligned} f_0(x, y, \ldots, \mu, \mu_x, \mu_y, \ldots, \mu_{xx}, \mu_{xy}, \ldots) \\ = \lim_{\varepsilon \to 0} f(\varepsilon, x, y, \ldots, \mu, \mu_x, \mu_y, \ldots) = 0. \end{aligned} \tag{2}$$

Clearly, Eq. (2) can only be expected to satisfy $m-n$ boundary conditions in x. Solving Eq. (2) subject to those m-n boundary conditions constitutes the outer problem.

Returning now to Eq. (1), we define a new independent variable τ by the transformation

$$\tau = \frac{x}{g(\varepsilon)}, \tag{3}$$

where g is an arbitrary scale function to be selected. Using Eq. (3), we may rewrite Eq. (1) as

$$f_i(\varepsilon, \tau, y, ..., \mu, \mu_\tau, \mu_y, ..., \mu_{\tau\tau}, \mu_{\tau y}, ...) = 0 \tag{4}$$

this being the so-called inner equation. Equation (4) is then to be solved subject to m boundary conditions in x, n of these being given by

$$\begin{aligned} \mu^0(x_f, y, ...) &= \mu^i(\tau_f, y, ...), \\ \mu^0(x_f, y, ...) &= \mu^i_\tau(\tau_f, y, ...), \\ &\vdots \end{aligned} \tag{5}$$

where

$$\tau_f = \frac{x_f}{g(\varepsilon)}, \tag{6}$$

and x_f is an arbitrary value of x. Clearly, τ_f is a function of the independent variables y,... . Solving Eq. (4) subject to the m-n boundary conditions and to Eq. (5) constitutes the inner problem.

The overall solution procedure then consists of the following steps:

(i) Solution of the outer problem.

(ii) Solution of the inner problem for arbitrary τ_f.

(iii) Solution of the inner problem for slightly different τ_f.

(iv) Verification of the proximity of successive iteration for the solution of the problem.

Several comments are now in order. The crux of the boundary value technique lies in the determination of the terminal boundary condition for the inner solution. This is achieved by an iterative process, in which an arbitrary point τ_f is chosen as the point where the inner solution ends and the outer solution begins. In more physical terms, τ_f may be interpreted as the thickness of the boundary layer region arising in the singular pertubation problem.

3. Application of the Boundary-Value Method to Viscous Fluid Flow

In order to evaluate the applicability of the boundary-value method to the Navier–Stokes equations, we have chosen two problems involving flows of a boundary layer nature and whose solutions via classical boundary layer theory have been obtained with some precision. These are uniform flow past a semi-infinite flat plate and flow with a linearly adverse external velocity. We consider these problems in turn.

(a) Uniform Flow Past a Semi-Infinite Flat Plate. Consider steady uniform two-dimensional incompressible viscous fluid flow past a semi-infinite plate. Using rectangular Cartesian coordinates with origin at the leading edge of the plate, the x—and y—coordinate axes parallel and perpendicular to the plate, respectively, the vorticity transfer equation may be written in terms of the stream function Ψ as

$$\Psi_y \Psi_{xyy} + \Psi_y \Psi_{xxx} - \Psi_x \Psi_{yyy} - \Psi_x \Psi_{xxy} = \varepsilon(\Psi_{xxxx} + 2\Psi_{xxyy} + \Psi_{yyyy}), \tag{7}$$

where ε is the inverse of the Reynolds numbers; i.e., $\varepsilon = 1/R = \nu/UL$. We seek a solution to Eq. (7) for small ε, subject to the no-penetration and no-slip boundary conditions at the plate; i.e.,

$$y = 0, \qquad x > 0; \ \Psi_x = \Psi_y = 0, \tag{8}$$

and to the uniform parallel flow boundary conditions faraway from the plate; i.e.,

$$y \to \infty, \qquad x > 0; \ \Psi_x = 0, \ \Psi_y = 1. \tag{9}$$

The outer flow equation, obtained by setting $\varepsilon = 0$, in Eq. (7), is given by

$$\Psi_y \Psi_{xyy} + \Psi_y \Psi_{xxx} - \Psi_x \Psi_{yyy} - \Psi_x \Psi_{xxy} = 0, \tag{10}$$

whose solution is required to satisfy the no-penetration boundary condition at the plate; i.e.,

$$y = 0, \qquad x > 0, \ \Psi_x = 0, \tag{11}$$

or equivalently and without any loss of generality

$$y = 0, \qquad x > 0, \ \Psi = 0. \tag{12}$$

The outer flow is also required to satisfy the uniform flow boundary conditions at infinity, given by Eq. (9). We note that as the outer Eq. (10) is of third-order and the equation of motion (7) is of fourth-order, the former can satisfy only three of the four boundary conditions given by Eqs. (8) and (9). Based on physical considerations, the no-slip boundary condition at the plate, given by Eq. (8), is abandoned in the outer flow problem. The solution of Eq. (10) subject to boundary conditions (9) and (12) is uniform flow; i.e.,

$$\Psi = y. \tag{13}$$

In order to obtain the inner flow equation, we adopt the following variable transformations

$$\tau = \frac{y}{\Delta(\varepsilon)}, \tag{14}$$

$$\Psi(x, y) = \Delta(\varepsilon)\,\chi(x, \tau), \tag{15}$$

where $\Delta(\varepsilon)$ is a yet to be determined function of ε. The function $\Delta(\varepsilon)$ is determined in the same manner as in the CMT; i.e.,

$$\Delta(\varepsilon) = \varepsilon^{1/2} = R^{-1/2}, \tag{16}$$

and then Eq. (7) may be written as

$$\chi_\tau \chi_{x\tau\tau} - \chi_x \chi_{\tau\tau\tau} + \varepsilon[\chi_\tau \chi_{xxx} - \chi_x \chi_{xx\tau}]$$
$$= \chi_{\tau\tau\tau\tau} + 2\varepsilon \chi_{xx} \chi_{\tau\tau} + \varepsilon^2 \chi_{xxxx}, \tag{17}$$

this being the inner flow equation, which is to be solved subject to the boundary conditions

$$\tau = 0, \qquad x > 0;\ \chi_x = \chi_\tau = 0, \tag{18}$$

$$\tau = \tau_f, \qquad x > 0;\ \chi_x = 0,\ \chi_\tau = 1, \tag{19}$$

which are obtained by transforming the boundary conditions (8) and (9), respectively.

As a first approximation, in the limit as ε tends to zero, or equivalently when R tends to infinity, Eq. (17) becomes.

$$\chi_\tau \chi_{x\tau\tau} = \chi_x \chi_{\tau\tau\tau} = \chi_{\tau\tau\tau\tau}, \tag{20}$$

subject to boundary conditions (18) and (19).

Using the similarity transformation

$$\chi(x, \tau) = x^{1/2} f(\eta), \tag{21}$$

$$\eta(x, \tau) = x^{1/2}\tau, \tag{22}$$

Eq. (20) becomes

$$f'f'' + f''' + 2f^{iv} = 0, \tag{23}$$

and the boundary conditions (18) and (19) become respectively,

$$\eta = 0, \qquad f = f' = 0, \tag{24}$$

$$\eta = \eta_f, \qquad f = \eta_f, f' = 1. \tag{25}$$

We note that Eq. (23) is the derivative of the Blassius equation of classical boundary layer theory. Furthermore, three of the boundary conditions (28) and (29) are derivatives of the corresponding boundary conditions of the Blassius problems, the exception being $(25)_1$ which cannot be prefixed in the Blassius problem as it is of third order. In fact, from the Blassius solution, we have

$$\eta = \eta_f, \qquad f = \eta_f - \beta,$$

where [5],

$$\beta = 1.7208.$$

This difference is due to the fact in the BVT as applied to this problem, the displacement effect is not taken into account as opposed to the solution obtained by matched asymptotic expansions (MAE). The two-point boundary-value problem given by (23)–(25) has been solved by orthogonal collocation for increasing values of η_f, and convergence is obtained at $\eta_f = 250$. The function f is shown in Table I, along with the Howarth solution of the Blassius problem [5]. It is seen that agreement is quite satisfactory, with a maximum percentage difference of 8 % between the two solutions.

(b) FLOW WITH A LINEARLY ADVERSE EXTERNAL VELOCITY. Consider the corresponding case to (a) when the fluid velocity profile faraway from the flat plate is given by

$$U = U_0 = Ax, \tag{26}$$

where U_0 and A are constants. Following the procedure shown in case (a), the equation of motion is given by (7), subject to the boundary conditions (8) and

$$y \to \infty, \qquad x > 0;\ \Psi_x = 0,\ \Psi_y = 1 - ax, \tag{27}$$

where $a = A/U_0$. As in case (a), the outer flow equation is given by (10),

TABLE I

Uniform Flow Past a Flat Plate: $U(x) = U_0$

η_f	10		100		250		Blassius			
η	$f(\eta)$	$f'(\eta)$	$f(\eta)$	$f'(\eta)$	$f(\eta)$	$f'(\eta)$	$f_{(exact)}$	$f'_{(exact)}$	$\%f$	$\%f'$
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.315183	0.608609	0.182794	0.362053	0.177615	0.353367	0.16557	0.32979	7.27	7.15
2.0	1.154978	1.030385	0.710180	0.682052	0.694724	0.669526	0.65003	0.62977	6.88	6.31
3.0	2.289017	1.198203	1.518564	0.917530	1.483860	0.88767	1.39682	0.84605	6.23	5.05
4.0	3.495045	1.197923	2.507571	1.041282	2.433052	0.992896	2.30576	0.95552	5.52	3.91
5.0	4.670909	1.152519	3.567016	1.064762	3.444793	1.022555	3.28329	0.99155	4.92	3.13
6.0	5.801540	1.110177	4.626192	1.054712	4.469697	1.025916	4.27964	0.99898	4.44	2.70
7.0	6.893947	1.075769	5.678636	1.050531	5.495788	1.025845	5.27926	0.99992	4.10	2.59
8.0	7.954936	1.046973	6.727265	1.046619	6.521067	1.024752	6.27923	1.00000	3.85	2.47
9.0	8.989178	1.022057	7.771878	1.042788	7.545400	1.023918	7.27029	1.00000	3.66	2.39
10.0	9.989178	1.000000	8.813518	1.040891	8.568825	1.022921	8.27929	1.00000	3.50	2.29
11.0	10.989178	1.000000	9.854043	1.039942	9.591351	1.022162	9.27929	1.00000	3.36	2.21
12.0	11.989178	1.000000	10.892929	1.037838	10.613153	1.021437	10.27929	1.00000	3.25	2.14
13.0	12.989178	1.000000	11.930001	1.036480	11.634237	1.020754	11.27929	1.00000	3.15	2.07
14.0	13.989178	1.000000	12.966112	1.035759	12.654718	1.020226	12.27929	1.00000	3.06	2.02

which is required to satisfy boundary condition (12) and the far flow boundary condition faraway from the plate, given by (27). The outer flow solution is given by

$$\Psi = (1 - ax)\, y. \tag{28}$$

The inner flow problem is formulated in a manner analogous to case (a), and the resulting problem in the limit as ε tends to zero can then be written as

$$\chi_\tau \chi_{x\tau\tau} - \chi_x \chi_{\tau\tau\tau} = \chi_{\tau\tau\tau\tau}, \tag{20}$$

$$\tau = 0, \qquad \chi_x = \chi_\tau = 0, \tag{18}$$

$$\tau = \tau_f, \qquad \chi_x = -a\tau_f,\ \chi_\tau = 1 - ax. \tag{29}$$

Utilizing the similarity transformation

$$\chi(x, \tau) = x^{1/2}\{f_0(\eta) - (8x)\, f_1(\eta) + (8x^*)\, f_2(\eta) - + \cdots\}, \tag{30}$$

$$\eta(x, \tau) = \frac{\tau x^{-1/2}}{2}, \tag{31}$$

$$x^*(x) = \frac{Ax}{U_0}, \tag{32}$$

in Eq. (22), and separating the terms of the same order in x, we obtain for the two lowest order equations, respectively

$$f'_0 f''_0 + f_0 f'''_0 + f^{iv} = 0, \tag{33}$$

$$-f_0 f''_1 + f'_1 f''_0 + 3 f'''_0 + f_1 + f_0 f'''_1 + f^{iv}_1 = 0. \tag{34}$$

The corresponding boundary conditions are given by

$$\eta = 0, \qquad f_0 = f'_0 = 0, \tag{35}$$

$$\eta = \eta_f, \qquad f_0 = 2\eta,\ f'_0 = 2, \tag{36}$$

$$\eta = 0, \qquad f_1 = f'_1 = 0, \tag{37}$$

$$\eta = \eta_f, \qquad f_1 = \frac{\eta}{4},\ f'_1 = \frac{1}{4}. \tag{38}$$

It is worth noting that (33) and (34) are derivatives of the Howarth lowest order pair of equations, respectively. These, being of third-order, cannot satisfy a *priori* all four boundary conditions (37) and (38). In fact,

TABLE II

Linearly Adverse External Velocity Flow: $U(x) = U_0 - Ax$

$\eta_f \rightarrow$	4		20		40		Howarth			
η	$f_1(\eta)$	$f'_1(\eta)$	$f_1(\eta)$	$f'_1(\eta)$	$f_1(\eta)$	$f'_1(\eta)$	$f_1(\eta)$	$f'_1(\eta)$	$\% f_1$	$\% f'_1$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.00000	0.00000	0	0
0.2	0.006901	0.063568	0.014935	0.143873	0.015436	0.148703	0.01908	0.18411	19.1	19.2
0.4	0.024850	0.118471	0.055294	0.254026	0.057150	0.262554	0.07098	0.32823	19.5	20.0
0.6	0.053315	0.164761	0.114308	0.330710	0.118146	0.341811	0.14771	0.43250	20.0	21.0
0.8	0.090196	0.202706	0.185535	0.376820	0.191763	0.389465	0.24137	0.49768	20.6	21.7
1.0	0.133894	0.233129	0.263317	0.397179	0.272154	0.410502	0.34432	0.52596	21.0	22.0
1.2	0.183061	0.257792	0.343092	0.397827	0.354605	0.411165	0.44959	0.53195	21.1	21.2
1.4	0.236831	0.279781	0.421570	0.385304	0.435713	0.398213	0.55143	0.49309	21.0	19.2
1.6	0.295029	0.302539	0.496759	0.365932	0.513420	0.378180	0.64581	0.44900	20.5	15.8
1.8	0.357896	0.325502	0.567849	0.345104	0.586889	0.356640	0.73069	0.39972	19.7	10.8
2.0	0.424593	0.339269	0.634958	0.326565	0.656239	0.337464	0.80592	0.35367	18.6	4.6
2.2	0.492826	0.341416	0.698727	0.311699	0.722134	0.322083	0.87273	0.31608	17.3	1.9
2.4	0.560549	0.334355	0.759778	0.298814	0.785217	0.308747	0.93305	0.28879	15.8	6.9
2.6	0.635989	0.318626	0.818039	0.282820	0.845412	0.292197	0.98889	0.27098	14.5	7.8
2.8	0.687626	0.297129	0.872674	0.263482	0.901856	0.272190	0.04192	0.26045	13.4	4.5
3.0	0.744790	0.297129	0.923575	0.246206	0.954436	0.254311	1.09338	0.25480	12.7	0.2
3.2	0.798035	0.274887	0.971509	0.234154	1.003945	0.241829	1.14403	0.25203	12.2	4.0
3.4	0.849243	0.259436	1.017658	0.228374	1.051603	0.235825	1.19430	0.25079	11.9	6.0
3.6	0.899744	0.253744	1.063224	0.228305	1.098671	0.235721	1.24440	0.25029	11.7	5.8
3.8	0.949946	0.251552	1.109251	0.232287	1.146172	0.239802	1.29443	0.25010	11.5	4.1
4.0	1.000000	0.250577	1.156277	0.238076	1.194716	0.245746	1.34444	0.25003	11.1	1.7
4.2	1.049950	0.250000	1.204446	0.243345	0.244433	0.251151	1.39444	0.25001	10.8	0.5
4.4	1.099764	0.248471	1.253452	0.246200	1.295008	0.254060	1.44445	0.25000	10.3	1.6

(38) is not prefixed in the Howarth solution, but rather is given from the solution as

$$\eta = \eta_f; \qquad f_1 = \frac{\eta}{4} + 0.3444. \tag{39}$$

This difference, as in the Blassius problem, is due to the fact that in the BVT solution, the displacement effect is neglected, while this is not the case in the MAE solution. The problem, given by (33), (35), and (36) and the problem given by (34), (37), and (38) have been solved by orthogonal collocation for decreasing values of η_f, convergence being obtained at $\eta_f = 250$ and $\eta_f = 40$, respectively. The function f_1, is shown in Table II, along with the solutions of Howarth.

4. Application of a Hybrid Boundary-Value-Coefficient Matching Method to Viscous Fluid Flow

In order to demonstrate the flexibility and the versatility of the BVT in the solution of fluid flow problems, we consider the classical problem of shear flow past a flat plate. Solutions have been obtained for this problem through the use of matched asymptotic expansions by Murray [6], and we solve this problem in two ways: first by application of the BVT on its own, and then by the joint application of the BVT along with matched asymptotic expansions.

Consider steady two-dimensional incompressible viscous fluid flow past a flat plate, where the fluid is in linear shear faraway from the plate. In rectangular Cartesian coordinates, the vorticity transfer equation is (7), which we wish to solve for small ε, subject to the boundary conditions (8) at the plate, and to linear shear flow faraway; i.e.,

$$y \to \infty, \qquad x > 0; \; \Psi_x = 0, \; \Psi_y = 1 + w_0 y. \tag{40}$$

As seen in Section 3, the outer flow equation is given by (10), which we now wish to solve subject to the boundary conditions at the plate given by (12), while the boundary condition (40) has to be satisfied faraway from the plate. The outer flow solution can then be immediately written as

$$\Psi = y + \frac{w_0 y^2}{2}. \tag{41}$$

The inner flow equation, as shown in Section 3, is given by (17), which we wish to solve subject to the boundary conditions (18) at the plate, and faraway from the plate, we have

$$\tau = \tau_f, \qquad x > 0;\ \chi_x = 0,\ \chi_\tau = 1 + w_0 y. \tag{42}$$

Using the expansion

$$\chi = \chi^0 + \varepsilon \chi^1, \tag{43}$$

in Eq. (17) and in boundary conditions (18) and (42), we obtain for χ^0

$$\chi^0_\tau \chi^0_{x\tau\tau} - \chi^0_x \chi^0_{\tau\tau\tau} = \chi^0_{\tau\tau\tau\tau}, \tag{44}$$

$$\tau = 0, \qquad \chi^0_\tau = \chi^0_x = 0, \tag{45}$$

$$\tau = \tau_f, \qquad \chi^0_\tau = 1,\ \chi^0_x = 0, \tag{46}$$

and for χ^1, we have

$$\chi^0_\tau \chi^1_{x\tau\tau} + \chi^0_{x\tau\tau}\chi^1 - \chi^0 \chi^1_{\tau\tau\tau} - \chi^0_{\tau\tau\tau}\chi^1_x = \chi^1_{\tau\tau\tau\tau}, \tag{47}$$

$$\tau = 0, \qquad \chi^1_\tau = \chi^1_x = 0, \tag{48}$$

$$\tau = \tau_f, \qquad \chi^1_\tau = w_0 t,\ \chi^1_x = 0. \tag{49}$$

Applying the transformations

$$\chi^0 = x^{1/2} f(\eta), \tag{50}$$

$$\chi^1 = 4 w_0 x g(\eta), \tag{51}$$

$$\eta = \frac{x^{1/2}}{2}, \tag{52}$$

the problems given by (44)–(45) and (47)–(49) become respectively

$$f^{iv} + f' f'' + f f''' = 0, \tag{53}$$

$$\eta = 0, f = f' = 0, \tag{54}$$

$$\eta = \eta_f, f = 2\eta, f' = 2, \tag{55}$$

and

$$g^{iv} + g' f'' + f g''' + 2 f''' g = 0, \tag{56}$$

$$\eta = 0,\ g = g' = 0, \tag{57}$$

$$\eta = \eta_f,\ g = \tfrac{1}{2}\eta^2,\ g' = \eta. \tag{58}$$

The problem given by (53)–(55) is equivalent to (23)–(25) and its solution is shown in Table I, and the problem given by (55)–(57) has been solved by orthogonal collocation and the results are displayed in Table III.

The above solution results from the direct application of the BVT to the problem of linear shear flow past a flat plate. With a view to evaluating the versatility of the BVT, we utilize this method jointly with that of matched asymptotic expansions (MAE) to solve this problem in an alternative manner. For this purpose, we introduce the asymptotic expansions for the inner and the outer stream functions.

$$\Psi(x,y) = (x,y) + \varepsilon^{1/2}\Psi^1(x,y) + \cdots , \tag{59}$$

$$\chi(x,\tau) = \varepsilon^{1/2}\chi^0(x,\tau) + \varepsilon\chi^1(x,\tau) + \cdots \tag{60}$$

Substituting Eq. (59) in the outer flow Eq. (10), and equating the coefficients of the various powers in ε to zero, we obtain for the zero-order outer equation

$$\nabla^2\Psi^0 = w_0, \tag{61}$$

where

$$w = w^0 + \varepsilon^{1/2}w^1 + \cdots , \tag{62}$$

subject to the boundary conditions

$$y = 0, \qquad x > 0, \ \Psi^0 = 0, \tag{63}$$

$$y \to \infty, \qquad x > 0, \ \Psi^0_y \to 1 + w_0 y. \tag{64}$$

The solution of Eq. (61) subject to boundary conditions (63) and (64) is given by

$$\Psi^0 = y + \frac{w_0 y^2}{2}. \tag{65}$$

Substitution of Eq. (65) in the inner flow Eq. (17) and equating the coefficients of the resulting power expansion in to zero, we have for the first-order inner equation

$$\chi^1_\tau\chi^1_{x\tau} - \chi^1_x\chi^1_{\tau\tau\tau} = \chi^1_{\tau\tau\tau\tau}. \tag{66}$$

Since Eq. (66) constitutes an exact differential, we have

$$\chi^1_\tau\chi^1_\tau - \chi^1_x\chi^1_{\tau\tau} = \chi^1_{\tau\tau\tau} + p_1(x), \tag{67}$$

TABLE III

Shear Flow Past a Flat Plate: $U(x) = U_0 + w_0 y$

η_f	4		13		17		Murray			
η	$g(\eta)$	$g'(\eta)$	$g(\eta)$	$g'(\eta)$	$g(\eta)$	$g'(\eta)$	g_{exact}	g'_{exact}	% g	% g'
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0
0.200000	0.015531	0.159806	0.023287	0.234236	0.014750	0.154011	0.06021	0.59050	75.50	73.92
0.400000	0.065669	0.345729	0.094107	0.474760	0.064156	0.346205	0.23139	1.10870	72.27	68.77
0.600000	0.155354	0.554576	0.213284	0.717130	0.155552	0.573142	0.49837	1.54698	68.79	62.95
0.800000	0.288707	0.781502	0.380827	0.957639	0.295370	0.829390	0.84423	1.89611	65.01	56.26
1.000000	0.468785	1.020875	0.595956	1.192403	0.488828	1.108420	1.25044	2.14999	60.91	40.45
1.200000	0.697525	1.267373	0.857182	1.418107	0.739820	1.403886	1.69800	2.31087	56.43	39.25
1.400000	0.975999	1.517870	1.162414	1.632076	1.051253	1.712142	2.16951	2.39299	51.54	28.45
1.600000	1.304779	1.769836	1.509106	1.832473	1.424998	2.024987	2.65164	2.42199	46.26	16.39
1.800000	1.683638	2.017113	1.894442	2.018482	1.860557	2.327314	3.13689	2.42963	40.69	4.21
2.000000	2.110397	2.245583	2.315563	2.190503	2.353921	2.599537	3.62408	2.44577	35.05	6.29
2.200000	2.578981	2.431047	2.769827	2.350338	2.896923	2.820021	4.11712	2.49144	26.64	13.19
2.400000	3.078963	2.559695	3.255108	2.501384	3.477109	2.969040	4.6236	2.57583	24.79	15.27
2.600000	3.599903	2.644530	3.770135	2.648819	4.079715	3.046559	5.15005	2.69786	20.78	12.93
2.800000	4.135753	2.714089	4.314867	2.799798	4.692622	3.077972	5.70444	2.85023	17.74	7.99
3.000000	4.686547	2.798345	4.890906	2.963632	5.310626	3.105475	6.29155	3.02377	15.59	2.70
3.200000	5.257936	2.927100	5.501683	3.147017	5.937873	3.178835	6.91480	3.21037	14.13	0.98
3.400000	5.862902	3.136255	6.150568	3.343321	6.587289	3.324842	7.57617	3.40417	13.05	2.33
3.600000	6.516895	3.411315	6.839391	3.544397	7.270156	3.507016	8.27670	3.60154	12.16	2.62
3.800000	7.228926	3.709266	7.568788	3.748453	7.990799	3.700052	9.01689	3.80052	11.38	2.64
4.000000	8.000000	4.000000	8.338647	3.949690	8.750414	3.896881	9.79695	4.00015	10.68	2.58
4.200000	8.829846	4.307734	9.148462	4.147936	9.549794	4.097511	10.61697	4.20004	10.05	2.44
4.400000	9.731920	4.752436	9.997626	4.343284	10.389481	4.299171	11.47697	4.40000	9.48	2.29

where $p_1(x)$ is an arbitrary function of x. Using the Matching Principle, we have

$$\chi_\tau^1(x, \infty) = 1, \tag{68}$$

$$p_1(x) = 0. \tag{69}$$

From boundary condition (8), we obtain

$$\chi_\tau^1(x, 0) = \chi_x^1(x, 0) = 0. \tag{70}$$

Utilizing the transformation

$$\chi^1(x, \tau) = x^{1/2} f(\eta), \tag{71}$$

$$\eta = \frac{x^{1/2}}{2} \tau, \tag{72}$$

we obtain from Eq. (67)

$$ff'' + f''' = 0, \tag{73}$$

and from boundary conditions (68) and (70), respectively

$$\eta = 0, \qquad f = f' = 0, \tag{74}$$

$$\eta = \eta_f, \qquad f' = 2 \tag{75}$$

Returning to the outer flow Eq. (10), the first-order external equation may be written as

$$\nabla^2 \Psi^1 = 0, \tag{76}$$

subject to the boundary conditions

$$y = 0, \qquad x < 0;\ \Psi^1 = 0, \tag{77}$$

$$y = 0, \qquad x > 0;\ \Psi^1 = -\beta x^{1/2}, \tag{78}$$

$$y \to \infty, \qquad x > 0,\ \Psi^1 = 0. \tag{79}$$

The boundary condition (78) is obtained by the Matching Principle. The solution of (76) subject to (77)–(79) is given by

$$\Psi^1 = -\beta \operatorname{Re} \sqrt{x + y^1}. \tag{80}$$

The first-order inner equation may be written as

$$\chi_\tau^0 \chi_{\tau\tau\tau}^1 + \chi_\tau^1 \chi_{x\tau}^0 - \chi_x^0 \chi_{\tau\tau\tau}^1 \chi_{x\tau\tau}^0 \chi^1 = \chi_{\tau\tau\tau\tau}^1. \tag{81}$$

Since Eq. (81) is an exact differential, we have

$$\chi^0_\tau \chi^1_{x\tau} + \chi^1_\tau \chi^0_{x\tau} - \chi^0_x \chi^1_{\tau\tau} - \chi^1_x \chi^0_{\tau\tau} = \chi^1_{\tau\tau\tau} + p_2(x), \tag{82}$$

where $p_2(x)$ is an arbitrary function of x. Matching yields

$$\chi^1_\tau(x, \infty) = \tau w_0, \tag{83}$$

$$p_2(x) = w_0 \chi^0_x(x, \infty) = \frac{-w_0}{2} \beta^{-1/2} x. \tag{84}$$

From the boundary condition (8), we have

$$\chi^1_x(x, 0) = \chi^1_y(x, 0) = 0. \tag{85}$$

Using the transformation

$$\chi^0(x, \tau) = x^{1/2} f(\eta), \tag{86}$$

$$\chi^1(x, \tau) = 4 w_0 x g(\eta), \tag{87}$$

$$\eta = \frac{x^{-1/2}\tau}{2}, \tag{88}$$

Eq. (82) becomes

$$g''' + fg'' - f'g' + 2f''g = -\beta, \tag{89}$$

and from boundary conditions (83) and (85), we have

$$\eta = 0, \qquad g = g' = 0, \tag{90}$$

$$\eta = \eta_f, \qquad g' = \eta_f. \tag{91}$$

The problem given by Eq. (89) subject to boundary conditions (90) and (91) is identical to the one fomulated and solved by Murray [6]. These results are also shown in Table III for the purpose of comparison. It is readily observed from Table III that the differences between the solution obtained directly from the BVT and that based on the BVT–MAE hybrid method or equivalently that of Murray, are significant, particularly if compared with our corresponding situation in the two former flow problems. Apart from the fact that the ordinary differential equations are different for the different problems under consideration and the consequent possible solution behaviour, we believe there are some important differences in the application of the BVT and the BVT–MAE method to this problem. Essentially, in the BVT, we have account for the external vorticity *only*, while in the BVT–MAE account has been taken of *both* the external vorticity *and* the displacement effect.

5. Concluding Remarks

The boundary-value technique, formulated and applied to singular perturbation problems of ordinary differential equations, has been extended to the solution of the Navier–Stokes equations for three standard high Reynolds number flows. These flows have been selected as solutions for they have been already obtained with confidence by classical boundary layer theory and the technique of matched asymptotic expansions, and hence can serve as an adequate test for the accuracy of the boundary value technique. In the following, we summarize our findings.

(i) The simplicity of the BVT, demonstrated by Roberts for ordinary differential equations, is shown to be valid for partial differential equations as well. Thus, the question of which asymptotic expansion to use which is inherent in all CMT's, simply does not arise in the BVT. However, in the transformation of variables, the scale function has still to be chosen *a priori* based on additional physico-mathematical arguments. This point is not touched upon by Roberts.

(ii) The flexibility of the BVT in that it can be used in a hybrid fashion with another technique, shown by Roberts for ordinary differential equations, is demonstrated in the problem of shear flow past a flat plate. On the other hand, when coupled to other techniques, the advantage of simplicity inherent in the BVT may be lost. This is demonstrated when the BVT is used in combination with the method of matched asymptotic expansions in the solution of the shear flow problem. Roberts does not appear to have recognized this limitation when applying the BVT in conjunction with other methods.

(iii) The fact that in the BVT the inner and the outer solutions are matched at a specific and definite location means that the question of the matching principle just does not arise. This advantage is shown in the three flows which have been determined by use if the BVT alone. Furthermore, the specific location at which the matching is performed provides a direct and precise measure of the boundary layer thickness.

(iv) From the analytical and the computational points of view, there is a substantial advantage in utilizing the BVT in that the inner flow is determined from the solution of a two-point boundary-value problem over a finite domain. This is in contrast to the CMT, for example the method of matched asymptotic expansions, where the inner flow is posed as a two-point boundary-value problem over an infinite domain. This feature is clearly demonstrated in the three flows considered, and this advantage should hold in general for flows in infinite regions.

(v) The mathematical question of convergence of the BVT remains open as noted by Roberts. Barring the settling of this question, we believe it necessary to extend the present investigation to more complex flows than those studied in the present work, albeit maintaining the approach of selecting flows for which reliable solutions have been obtained by CMT's and other established methods. In the way of examples, we mention low Eckman number, rotating flows, corner boundary layers, and trailing edge flows.

(vi) The criterion which Roberts utilized to terminate the iteration, which has also been used in the present work, depends on the matching of the inner and outer flow variables only and nothing is said about the derivatives. Clearly, better matching should be obtained if a more complete criterion is envisaged, involving the first few derivatives, up to the order of the equation say, as well as the flow variables themselves.

(vii) When applied to partial differential equations, the joining of the inner and the outer solutions is performed along a curve or surface as the case may be, due to the increase in dimensionality of the problem as compared to the case of ordinary differential equations.

(viii) As advanced by Roberts, the BVT is applicable only to singular perturbation problems of the type where the small parameter multiplies the highest derivative. Then there is clearly room for extending the method to other types of singularities, particularly if we wish to apply the method to low Reynolds numbers flows amongst others.

(ix) As Roberts has stated, the question of which boundary condition to drop in the outer flow problem still has to be based on physical arguments as is the case in coefficient matching methods.

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