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Convective Eigenvalue Problems for Convergence Enhancement of Eigenfunction Expansions in Convection–Diffusion Problems

The present work considers the application of the generalized integral transform technique (GITT) in the solution of a class of linear or nonlinear convection–diffusion problems, by fully or partially incorporating the convective effects into the chosen eigenvalue problem that forms the basis of the proposed eigenfunction expansion. The aim is to improve convergence behavior of the eigenfunction expansions, especially in the case of formulations with significant convective effects, by simultaneously accounting for the relative importance of convective and diffusive effects within the eigenfunctions themselves, in comparison against the more traditional GITT solution path, which adopts a purely diffusive eigenvalue problem, and the convective effects are fully incorporated into the problem source term. After identifying a characteristic convective operator, and through a straightforward algebraic transformation of the original convection–diffusion problem, basically by redefining the coefficients associated with the transient and diffusive terms, the characteristic convective term is merged into a generalized diffusion operator with a space-variable diffusion coefficient. The generalized diffusion problem then naturally leads to the eigenvalue problem to be chosen in proposing the eigenfunction expansion for the linear situation, as well as for the appropriate linearized version in the case of a nonlinear application. The resulting eigenvalue problem with space variable coefficients is then solved through the GITT itself, yielding the corresponding algebraic eigenvalue problem, upon selection of a simple auxiliary eigenvalue problem of known analytical solution. The GITT is also employed in the solution of the generalized diffusion problem, and the resulting transformed ordinary differential equations (ODE) system is solved either analytically, for the linear case, or numerically, for the general nonlinear formulation. The developed methodology is illustrated for linear and nonlinear applications, both in one-dimensional (1D) and multidimensional formulations, as represented by test cases based on Burgers' equation. [DOI: 10.1115/1.4037576]

Introduction

Despite all of the progress achieved on numerical methods and computer software for heat and fluid flow analysis, the classical analytical methods remain of great interest in thermal engineering and related applications, going much beyond their usefulness as benchmarking tools for numerical implementations and commercial codes. Exact or approximate analytical solutions to partial differential equations in transport phenomena play a major role in offering robust and cost-effective simulations in a number of applied problems and design tasks. For instance, one such approach is the integral transform method, a well-known mathematical tool for solving various classes of linear transformable diffusion problems, which is documented in classical treatises [1–6]. In the remarkable book by Mikhailov and Özisik [6], a fairly large amount of analytical work on integral transforms is unified and consolidated as exact solutions for seven general classes of diffusion problems, including different possibilities of coupling through either the equations or boundary conditions. The analytical solutions systematically obtained and compiled in Ref. [6] are directly applicable to various problems in heat conduction, simultaneous heat and mass transfer, transpiration cooling of porous media, unsteady flow through conduits, forced convection heat transfer in channels, drying of porous moist material, heat or mass diffusion in heterogeneous media, heat or mass diffusion into a body from a limited volume of well-stirred fluid, diffusion with reversible and irreversible chemical reactions, etc. In all such applications, the availability of a ready-to-use analytical solution for computational evaluation is unbeatable in terms of accuracy, robustness and computational speed. In all such archival contributions, the methodology leads to the consideration of a diffusive eigenvalue problem, which offers the basis for the eigenfunction expansions, usually in the form of Sturm–Liouville type problems.

For the last three decades, under the stimulus of constructing a hybrid numerical–analytical approach for a more flexible treatment of nontransformable problems, such as in the case of nonlinear and/or convection–diffusion formulations, the so-called generalized integral transform technique (GITT) was progressively and successfully advanced [7–14], becoming a well-established alternative hybrid tool in the analysis of different classes of heat and fluid flow problems. Among the various applications that have been dealt with, one may point out the analysis of heat conduction with temperature-dependent thermophysical properties, mass transfer in metal oxidation at high temperatures, heat conduction in ablative thermal protections, double-pipe heat exchanger analysis, forced convection in irregularly shaped channels, mass transfer in tracer analysis of petroleum reservoirs, convective heat transfer in nanofluids with temperature-dependent thermophysical properties, mass diffusion in fusion reactor walls, migration of radionuclides in soils from waste of uranium mining,

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aerodynamic heating in space vehicles during atmospheric reentry, conjugated heat transfer in microheat exchangers, anti-icing thermal systems for aeronautical Pitot probes, analysis of biodiesel synthesis in microreactors, etc., as extensively reviewed in Refs. [8–14]. In the GITT approach, practically as a first step, an eigenvalue problem is chosen that incorporates sufficient information on the originally formulated problem, yielding an appropriate basis for the eigenfunction expansions proposition. Usually, only a linearized diffusive formulation is accounted for in the eigenvalue problem choice, reflecting as much as possible the original equation and boundary operators, which may be itself handled by the GITT for space variable coefficients in any general functional form [8,15–17]. The first contributions on the GITT for transient nonlinear convection–diffusion problems [18–20] already reflect this preference, while retaining only diffusive operators in the eigenfunctions construction, thus avoiding nonclassical eigenvalue problems that incorporate the convective terms information. In problems with more significant convective effects, this solution path may lead to slower convergence rates in the eigenfunction expansions. Such difficulties have been overcome through the use of convergence acceleration schemes, such as with analytical filtering and integral balance procedures [11,21,22], or by adopting a partial integral transformation scheme, which transforms only those spatial variables in which the diffusive effects predominate, yielding a transformed partial differential system to be numerically solved [23–27].

The present work addresses the development of a variant of the more traditional and well-established GITT approach, through the incorporation of convective effects within the eigenvalue problem formulation, aimed at achieving improved convergence rates for transient convection–diffusion problems with significant convective effects. For this purpose, a straightforward algebraic coefficients exponential transformation is introduced, which essentially rewrites the original convection–diffusion problem formulation, so as to fully or partially transfer the information on the convective terms, into a new generalized diffusion operator. This algebraic transformation is quite common and appears in different contexts, such as in obtaining the exact solution of transpiration cooling in a porous slab for constant and uniform velocity [6]. Then, the usual choice of a diffusive eigenvalue problem in the construction of the eigenfunction expansions will naturally lead to the consideration of a generalized diffusive eigenvalue problem, which incorporates the convective effects via the modified space variable coefficients. In fact, the solution path can be interpreted as equivalent to directly considering a convective eigenvalue problem, and promoting the coefficient exponential transformation while handling its solution. Either way, the eigenvalue problem that includes the convective terms effects can, in principle, be solved through the GITT itself [8,15–17], by considering a sufficiently simple auxiliary eigenvalue problem, and reducing the chosen eigenvalue problem to the solution of an algebraic matrix eigenvalue problem.

The proposed approach is here illustrated by considering both one-dimensional (1D) and two-dimensional (2D) test cases based on the classical Burgers equation for linear and nonlinear transient convection–diffusion. The improved convergence rates are then observed, for significant magnitude of the convective effects, and critically compared against results obtained through the same method, but under the traditional diffusive eigenvalue problem choice. The aim is to demonstrate that the convective eigenvalue problem basis is a simple alternative of the integral transform method that can be readily employed in several applications involving simultaneous convection and diffusion effects, with relative advantages over the more traditional approach with purely diffusive eigenvalue problems.

Analysis

The adoption of convective eigenvalue problems in the integral transforms solution of transient convection–diffusion is here analyzed, first by considering a fairly general nonlinear one-dimensional formulation that should allow for a basic understanding of the proposed reformulation of the convection–diffusion problem as a generalized diffusion problem. Then, the extension to multidimensional situations is pursued, again considering a general nonlinear formulation. Finally, both linear and nonlinear special cases are exploited, which will provide the computational test cases here selected for demonstration of the approach.

Formal Generalized Integral Transform Technique Solution: Diffusive Eigenvalue Problem. Before advancing to the proposed novel approach, the GITT [7–14], with the usual diffusive eigenvalue problem choice, is summarized, as employed in solving the general one-dimensional nonlinear convection–diffusion problem given by

$$\breve{w}(x,t,T)\,\frac{\partial T(x,t)}{\partial t}+\breve{u}\,(x,t,T)\,\frac{\partial T(x,t)}{\partial x}=\frac{\partial}{\partial x}\left[\breve{k}(x,t,T)\,\frac{\partial T(x,t)}{\partial x}\right]-\breve{d}(x,t,T)\,T(x,t)+\breve{P}(x,t,T),\ \text{in}\ x_0<x<x_1,\ t>0 \tag{1a}$$

with the following initial and boundary conditions:

$$T(x,0)=f(x),\quad \text{in}\ \ x_0\le x\le x_1 \tag{1b}$$

$$\breve{\alpha}_0(t,T)T(x_0,t)-\breve{\beta}_0(t,T)\breve{k}(x_0,t,T)\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x_0}=\breve{\phi}_0(t,T),\quad t>0 \tag{1c}$$

$$\breve{\alpha}_1(t,T)T(x_1,t)+\breve{\beta}_1(t,T)\breve{k}(x_1,t,T)\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x_1}=\breve{\phi}_1(t,T),\quad t>0 \tag{1d}$$

where all the coefficients and the source term of the proposed equation and boundary conditions were allowed to be nonlinear, and the explicit general dependence on the space coordinate and time variable is also considered.

Following the ideas in the GITT approach [7–14], the next step would be rewriting problem (1) by considering characteristic linear coefficients for the transient, diffusion, and dissipation operators, given in the form of space variable only coefficients ($w(x)$, $k(x)$, and $d(x)$), in substitution to the original general nonlinear coefficients ($\breve{w}\,(x,t,T)$, $\breve{k}\,(x,t,T)$, and $\breve{d}\,(x,t,T)$). Then, the source term is modified accordingly, to incorporate the nonlinear convective term and the differences between the nonlinear and characteristic transient, diffusion, and dissipation terms. The same rearrangement is implemented in the boundary conditions, by considering linear characteristic constant values for the boundary coefficients, α_0, β_0, α_1, and β_1, in place of the general nonlinear coefficients $\breve{\alpha}_0(t,T)$, $\breve{\beta}_0(t,T)$, $\breve{\alpha}_1(t,T)$, and $\breve{\beta}_1(t,T)$, and again the difference is merged into the redefined boundary condition source terms. These chosen characteristic coefficients, in both equation and boundary conditions, already imply the choice of the diffusive eigenvalue problem to be considered in constructing the eigenfunction expansion basis, following the usual GITT solution procedure. Therefore, problem (1) can be rewritten, without loss of generality, in the form

$$w(x)\,\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[k(x)\,\frac{\partial T(x,t)}{\partial x}\right]-d(x)T(x,t)+P(x,t,T),\quad x_0<x<x_1,\quad t>0 \tag{2a}$$

with the following initial and boundary conditions:

$$T(x,0) = f(x), \quad \text{in } x_0 \le x \le x_1 \tag{2b}$$

$$\alpha_0 T(x_0,t) - \beta_0 k(x_0)\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x_0} = \phi_0(t,T), \quad t>0 \tag{2c}$$

$$\alpha_1 T(x_1,t) + \beta_1 k(x_1)\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x_1} = \phi_1(t,T), \quad t>0 \tag{2d}$$

The redefined source terms are then given by

$$\begin{aligned} P(x,t,T) = {} & \left[w(x) - \breve{w}(x,t,T)\right]\frac{\partial T(x,t)}{\partial t} - \breve{u}(x,t,T)\frac{\partial T(x,t)}{\partial x} \\ & - \frac{\partial}{\partial x}\left[\left(k(x) - \breve{k}(x,t,T)\right)\frac{\partial T(x,t)}{\partial x}\right] \\ & + \left[d(x) - \breve{d}(x,t,T)\right]T(x,t) + \breve{P}(x,t,T) \end{aligned} \tag{3a}$$

$$\begin{aligned} \phi_0(t,T) = {} & \left[\alpha_0 - \breve{\alpha}_0(t,T)\right]T(x_0,t) \\ & - \left[\beta_0 k(x_0) - \breve{\beta}_0(t,T)\breve{k}(x_0,t,T)\right]\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x_0} \\ & + \breve{\phi}_0(t,T) \end{aligned} \tag{3b}$$

$$\begin{aligned} \phi_1(t,T) = {} & \left[\alpha_1 - \breve{\alpha}_1(t,T)\right]T(x_1,t) \\ & + \left[\beta_1 k(x_1) - \breve{\beta}_1(t,T)\breve{k}(x_1,t,T)\right]\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x_1} \\ & + \breve{\phi}_1(t,T) \end{aligned} \tag{3c}$$

Therefore, the following diffusive eigenvalue problem is adopted in constructing the eigenfunction expansion:

$$\frac{d}{dx}\left[k(x)\frac{d\psi(x)}{dx}\right] + \left[\mu^2 w(x) - d(x)\right]\psi(x) = 0, \quad x_0 < x < x_1 \tag{4a}$$

with boundary conditions

$$\alpha_0\psi(x_0) - \beta_0 k(x_0)\frac{d\psi(x)}{dx}\bigg|_{x=x_0} = 0 \tag{4b}$$

$$\alpha_1\psi(x_1) + \beta_1 k(x_1)\frac{d\psi(x)}{dx}\bigg|_{x=x_1} = 0 \tag{4c}$$

This Sturm–Liouville problem then leads to the development of the following integral transform pair:

$$T(x,t) = \sum_{i=1}^{\infty}\frac{1}{N_i}\psi_i(x)\bar{T}_i(t), \quad \text{inverse} \tag{5a}$$

$$\bar{T}_i(t) = \int_{x_0}^{x_1} w(x)\psi_i(x)T(x,t)dx, \quad \text{transform} \tag{5b}$$

where the normalization integral is given by

$$N_i = \int_{x_0}^{x_1} w(x)\psi_i^2(x)dx \tag{5c}$$

The integral transformation process itself is accomplished through the operator $\int_{x_0}^{x_1}\psi_i(x)\_dx$, which after manipulation of the boundary conditions from both the original partial differential equation and the eigenvalue problem, leads to the following transformed ordinary differential equations (ODE) system [7–14]:

$$\frac{d\bar{T}_i(t)}{dt} + \mu_i^2\bar{T}_i(t) = \bar{g}_i(t,\bar{\mathbf{T}}), \quad t>0, \quad i = 1,2,3... \tag{6a}$$

where

$$\bar{\mathbf{T}} = \{\bar{T}_1(t), \bar{T}_2(t), \bar{T}_3(t), ..., \bar{T}_j(t), ...\} \tag{6b}$$

with transformed initial condition obtained through the operator $\int_{x_0}^{x_1} w(x)\psi_i(x)\_dx$

$$\bar{T}_i(0) = \bar{f}_i \equiv \int_{x_0}^{x_1} w(x)\psi_i(x)f(x)dx \tag{6c}$$

where the transformed coupling source term is given by

$$\begin{aligned} \bar{g}_i(t,\bar{\mathbf{T}}) = {} & \int_{x_0}^{x_1}\psi_i(x)P(x,t,T)dx + \phi_0(t,T(x_0,t)) \\ & \times\left[\frac{\psi(x_0) + k(x_0)\frac{d\psi(x_0)}{dx}}{\alpha_0 + \beta_0}\right] + \phi_1(t,T(x_1,t)) \\ & \times\left[\frac{\psi(x_1) - k(x_1)\frac{d\psi(x_1)}{dx}}{\alpha_1 + \beta_1}\right] \end{aligned} \tag{6d}$$

Numerical solution of the ODE's system (6) for the general nonlinear situation here considered can be accomplished by well-tested routines for initial value problems with automatic accuracy control [28], and then provide results for the transformed potentials along the t variable. However, it is well known that improved convergence is achieved by appropriate filtering of problem (2), particularly to reduce the importance of nonhomogeneous boundary conditions [7–14], and even the employment of eigenvalue problems with nonlinear boundary conditions, as more recently advanced [29]. In any case, it is clear from the above development that the convective term in the original formulation of problem (1) is never accounted for, even partially, in the proposed eigenvalue problem, and is directly transported to the modified nonlinear source term. Therefore, for an increased importance of the convective term in comparison to the diffusion effects, a decrease on the convergence rate of the eigenfunction expansion above derived can be expected since the convection effect is only present in the source term, which in fact contributes to slowing down the expansion convergence rate.

Coefficient Transformation: Convective Eigenvalue Problem. Starting with the general one-dimensional nonlinear problem (1), it will be first shown how the convection–diffusion problem can be rewritten so as to bring up a generalized diffusive formulation that shall be useful in defining an alternative eigenvalue problem so as to incorporate part or all of the convective information from the original problem. For convenience and without loss of generality, after expanding the diffusion operator, Eq. (1a) is rewritten as

$$\begin{aligned} & w^*(x,t,T)\frac{\partial T(x,t)}{\partial t} + u^*(x,t,T)\frac{\partial T(x,t)}{\partial x} \\ & = \frac{\partial^2 T(x,t)}{\partial x^2} - d^*(x,t,T)T(x,t) \\ & + P^*(x,t,T), \quad x_0 < x < x_1, \quad t>0 \end{aligned} \tag{7}$$

where

$$w^*(x,t,T) = \frac{\breve{w}(x,t,T)}{\breve{k}(x,t,T)} \tag{8a}$$

$$u^*(x,t,T) = \frac{1}{\breve{k}\,(x,t,T)}\left[\breve{u}\,(x,t,T) - \frac{\partial \breve{k}\,(x,t,T)}{\partial x}\right] \qquad (8b)$$

$$d^*(x,t,T) = \frac{\breve{d}\,(x,t,T)}{\breve{k}\,(x,t,T)} \qquad (8c)$$

$$P^*(x,t,T) = \frac{\breve{P}\,(x,t,T)}{\breve{k}\,(x,t,T)} \qquad (8d)$$

By employing an exponential factor $\hat{k}(x,t,T) = e^{-\int u^* dx}$ throughout Eq. (7), it can be readily shown that problem (1) can be rewritten as a generalized diffusion problem in the form

$$\hat{w}(x,t,T)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\hat{k}(x,t,T)\frac{\partial T(x,t)}{\partial x}\right] - \hat{d}(x,t,T)T(x,t) + \hat{P}(x,t,T), \quad x_0 < x < x_1, t > 0 \qquad (9)$$

where

$$\hat{w}(x,t,T) = w^*(x,t,T)\,\hat{k}(x,t,T) \qquad (10a)$$

$$\hat{d}(x,t,T) = d^*(x,t,T)\,\hat{k}(x,t,T) \qquad (10b)$$

$$\hat{P}(x,t,T) = P^*(x,t,T)\,\hat{k}(x,t,T) \qquad (10c)$$

$$\hat{k}(x,t,T) = e^{-\int u^* dx} \qquad (10d)$$

Although the analysis of integral transforms with nonlinear eigenvalue problems [29] is out of the scope of the present contribution, it is instructive to verify that the convection–diffusion problem given in Eq. (1), for any general form of the associated coefficients and source term, can be actually rewritten as a generalized diffusion problem with modified spatially variable coefficients.

Now, one may return to the traditional solution path through the GITT for nonlinear problems [7–14], as discussed in the Formal GITT Solution: Diffusive Eigenvalue Problem section, which first involves the selection of characteristic space variable coefficients to replace the nonlinear ones in the general formulation, while gathering the remaining nonlinear terms into the nonlinear source functions. Thus, when applying the GITT, Eq. (9) would then be rewritten as

$$w(x)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[k(x)\frac{\partial T(x,t)}{\partial x}\right] - d(x)T(x,t) + P(x,t,T), \quad x_0 < x < x_1,\ t > 0 \qquad (11)$$

where the newly chosen coefficients $w(x)$, $k(x)$, and $d(x)$ are linear characteristic representations of the modified nonlinear operators $\hat{w}(x,t,T)$, $\hat{k}(x,t,T)$, and $\hat{d}(x,t,T)$, while the nonlinearities are collapsed into the nonlinear source term, $P(x,t,T)$, at this point given by

$$P(x,t,T) = \left[w(x) - \hat{w}(x,t,T)\right]\frac{\partial T(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[\left(k(x) - \hat{k}(x,t,T)\right)\frac{\partial T(x,t)}{\partial x}\right] + \left[d(x) - \hat{d}(x,t,T)\right]T(x,t) + \hat{P}(x,t,T) \qquad (12)$$

with the boundary conditions coefficients and source terms also rewritten appropriately. Now, the selected characteristic linear coefficients are responsible for at least partially accounting for the convection term influence within the chosen eigenvalue problem. The remaining steps in the GITT approach are essentially the same as already described.

An equivalent way of proposing the incorporation of the convective effects into the eigenvalue problem would be to consider a characteristic linear convection coefficient, ahead of performing the coefficients exponential transformation, thus offering a slightly modified version of Eq. (2a). In this case, the following equivalent formulation of Eq. (1a) may be adopted, instead of Eq. (2a):

$$w(x)\frac{\partial T(x,t)}{\partial t} + u(x)\frac{\partial T(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[k(x)\frac{\partial T(x,t)}{\partial x}\right] - d(x)T(x,t) + P(x,t,T), \quad x_0 < x < x_1,\ t > 0 \qquad (13a)$$

with the modified source term

$$P(x,t,T) = \left[w(x) - \breve{w}\,(x,t,T)\right]\frac{\partial T(x,t)}{\partial t} + \left[u(x) - \breve{u}\,(x,t,T)\right]\frac{\partial T(x,t)}{\partial x} - \frac{\partial}{\partial x}\left[\left(k(x) - \breve{k}\,(x,t,T)\right)\frac{\partial T(x,t)}{\partial x}\right] + \left[d(x) - \breve{d}\,(x,t,T)\right]T(x,t) + \breve{P}\,(x,t,T) \qquad (13b)$$

where $u(x)$ is a characteristic linear representation of the convective term coefficient, while the remaining of the nonlinear convective operator is incorporated into the nonlinear source term, $P(x,t,T)$. Equation (13a) is similarly modified to the diffusive representation of Eq. (9), which, after the coefficients exponential transformation, is written as a generalized diffusion problem in the form

$$\hat{w}(x)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\hat{k}(x)\frac{\partial T(x,t)}{\partial x}\right] - \hat{d}(x)T(x,t) + \hat{P}(x,t,T), \quad x_0 < x < x_1, t > 0 \qquad (14a)$$

where

$$\hat{w}(x) = w(x)\,\hat{k}(x)/k(x) \qquad (14b)$$

$$\hat{d}(x) = d(x)\,\hat{k}(x)/k(x) \qquad (14c)$$

$$\hat{P}(x,t,T) = P(x,t,T)\,\hat{k}(x)/k(x) \qquad (14d)$$

$$u^*(x) = \frac{1}{k(x)}\left[u(x) - \frac{dk(x)}{dx}\right]; \quad \text{and} \quad \hat{k}(x) = e^{-\int u^*(x)dx} \qquad (14e)$$

Equation (14a) is now a special case of the nonlinear diffusion problem representation that has been extensively handled through the GITT [7–14], as summarized in the Formal GITT Solution: Diffusive Eigenvalue Problem section, but is here handled through an eigenfunction expansion basis that includes convective effects through the characteristic convective term coefficient, $u(x)$, present in the generalized diffusion coefficient $\hat{k}(x)$. The self-adjoint eigenvalue problem with space variable coefficients to be considered would then be given by the following equation:

$$\frac{d}{dx}\left[\hat{k}(x)\frac{d\psi(x)}{dx}\right] + \left[\mu^2\hat{w}(x) - \hat{d}(x)\right]\psi(x) = 0, \quad x_0 < x < x_1 \qquad (15)$$

which can be readily solved by the GITT itself [8,15–17], yielding the corresponding algebraic eigenvalue problem.

Convective Eigenvalue Problem: Multidimensional Formulation. The same straightforward transformation can be employed to rewrite a more general multidimensional convection–diffusion formulation, given in the general region V with the position vector $\mathbf{x}$. Skipping the derivation of the transformation for the case with nonlinear coefficients, such as in Eq. (1), the following nonlinear multidimensional problem, already with the linear characteristic coefficients identified, is considered:

$$w(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}+\mathbf{u}(\mathbf{x}).\nabla T(\mathbf{x},t) = \nabla.\left[k(\mathbf{x})\nabla T(\mathbf{x},t)\right]-d(\mathbf{x})T(\mathbf{x},t) + P(\mathbf{x},t,T),\quad \mathbf{x}\in V,\ t>0 \tag{16}$$

where the linear coefficients in each operator, dependent only on the spatial variables, already imply the choice of characteristic functional behaviors to be accounted for in the eigenfunction expansion basis, while the remaining nonlinearities are gathered into the redefined nonlinear source term $P(\mathbf{x},t,T)$.

Considering that the convective term coefficient vector $\mathbf{u}$ can be represented in the three-dimensional situation by the three components $\{u_x, u_y, u_z\}$, here illustrating the transformation in the Cartesian coordinates system, $\mathbf{x}=\{x,y,z\}$, then Eq. (16) is rewritten in the generalized diffusive form as

$$\hat{w}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}=\hat{k}_y(\mathbf{x})\hat{k}_z(\mathbf{x})\frac{\partial}{\partial x}\left[\hat{k}_x(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x}\right] + \hat{k}_x(\mathbf{x})\hat{k}_z(\mathbf{x})\frac{\partial}{\partial y}\left[\hat{k}_y(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial y}\right] + \hat{k}_x(\mathbf{x})\hat{k}_y(\mathbf{x})\frac{\partial}{\partial z}\left[\hat{k}_z(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial z}\right] - \hat{d}(\mathbf{x})T(\mathbf{x},t)+\hat{P}(\mathbf{x},t,T),\quad \mathbf{x}\in V,\ t>0 \tag{17a}$$

where

$$\hat{k}(\mathbf{x})=\hat{k}_x(\mathbf{x})\hat{k}_y(\mathbf{x})\hat{k}_z(\mathbf{x}) \tag{17b}$$

$$\hat{w}(\mathbf{x})=w(\mathbf{x})\,\hat{k}(\mathbf{x})/k(\mathbf{x}) \tag{17c}$$

$$\hat{d}(\mathbf{x})=d(\mathbf{x})\,\hat{k}(\mathbf{x})/k(\mathbf{x}) \tag{17d}$$

$$\hat{P}(\mathbf{x},t,T)=P(\mathbf{x},t,T)\,\hat{k}(\mathbf{x})/k(\mathbf{x}) \tag{17e}$$

$$\mathbf{u}^*(\mathbf{x})=\frac{1}{k(\mathbf{x})}\left[\mathbf{u}(\mathbf{x})-\nabla k(\mathbf{x})\right] \tag{17f}$$

$$\hat{k}_x(\mathbf{x})=e^{-\int u_x^*(\mathbf{x})dx} \tag{17g}$$

$$\hat{k}_y(\mathbf{x})=e^{-\int u_y^*(\mathbf{x})dy} \tag{17h}$$

$$\hat{k}_z(\mathbf{x})=e^{-\int u_z^*(\mathbf{x})dz} \tag{17i}$$

For the general situation of Eq. (17a), the separation of variables method, as applied to the homogeneous version of the problem, leads to a nonself-adjoint eigenvalue problem; hence, the eigenfunctions are not orthogonal and the classical integral transformation approach is not directly applicable in the present form. However, the GITT [7–14] can still be directly employed with an appropriate choice of a self-adjoint eigenvalue problem. In addition, a bi-orthogonal set might still be obtained that yields a transformed system with coupling only at the nonlinear source terms [30]. On the other hand, when the transformed diffusion coefficients are functions of only the corresponding space coordinate, or $\hat{k}_x(\mathbf{x})=\hat{k}_x(x),\ \hat{k}_y(\mathbf{x})=\hat{k}_y(y);\ \hat{k}_z(\mathbf{x})=\hat{k}_z(z)$, with the consequent restrictions on the choices of the characteristic linear coefficients $k(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$, a modified diffusion formulation is constructed, which leads to a self-adjoint eigenvalue problem and can be written in such special case as

$$\hat{w}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t}=\nabla.\left[\hat{k}(\mathbf{x})\nabla T(\mathbf{x},t)\right]-\hat{d}(\mathbf{x})T(\mathbf{x},t) + \hat{P}(\mathbf{x},t,T),\quad \mathbf{x}\in V,\ t>0 \tag{18a}$$

where

$$\hat{k}(\mathbf{x})=\hat{k}_x(x)\hat{k}_y(y)\hat{k}_z(z) \tag{18b}$$

for which the appropriate self-adjoint eigenvalue problem would be

$$\nabla.[\hat{k}(\mathbf{x})\nabla\psi(\mathbf{x})]+[\mu^2\hat{w}(\mathbf{x})-\hat{d}(\mathbf{x})]\psi(\mathbf{x})=0,\quad \mathbf{x}\in V \tag{19}$$

which can be directly solved by the GITT itself [15–17] to yield the corresponding algebraic eigenvalue problem for the corresponding eigenvalues and eigenvectors that reconstruct the desired eigenfunction, $\psi(\mathbf{x})$. Problem (19) again incorporates relevant information on the convective effects, as specified in the chosen linear convective term coefficients that undergo the exponential transformation, which can provide a desirable convergence enhancement effect in the integral transform solution of this multidimensional convection–diffusion problem. The remaining steps in the GITT solution of both Eqs. (18a) and (19) are well documented in the pertinent literature [7–14] and are not repeated here.

Test Cases

One-Dimensional Burgers Equation. The illustration of the proposed procedure is started with the analysis of the one-dimensional Burgers equation, both in linear and nonlinear formulations [8,18], which allows for the analysis of the convective eigenvalue problem choice on the eigenfunction expansion convergence behavior. The convection–diffusion problem here first analyzed is given by

$$\frac{\partial T(x,t)}{\partial t}+U(T)\frac{\partial T(x,t)}{\partial x}=\frac{\partial^2 T(x,t)}{\partial x^2},\quad 0<x<1,\ t>0 \tag{20a}$$

$$T(x,0)=1 \tag{20b}$$

$$T(0,t)=0,\ t>0 \tag{20c}$$

$$T(1,t)=0,\ t>0 \tag{20d}$$

where the nonlinear velocity coefficient is taken as

$$U(T)=u_0+b.T \tag{20e}$$

which readily allows for the separate analysis of the linear and nonlinear situations.

Although in this test case, the boundary conditions were considered to be homogeneous, any nonhomogeneous boundary condition (first, second, or third kind) can be readily handled, with the observation that for improved convergence, it is always recommended to reduce the importance of the boundary source terms, which can be achieved via the proposition of a filtering solution [11].

From direct comparison to the general one-dimensional formulation in Eq. (7), choosing the characteristic convective coefficient as the linear portion of Eq. (20e), and moving the remaining nonlinear convective term to the source term, the following correspondence is proposed:

$$w(x) = 1 \tag{21a}$$

$$u(x) = u_0 \tag{21b}$$

$$k(x) = 1 \tag{21c}$$

$$d(x) = 0 \tag{21d}$$

$$P(x,t,T) = -bT(x,t)\frac{\partial T}{\partial x} \tag{21e}$$

where for the special case of a linear situation ($b = 0$), the nonlinear source term vanishes. For this particular linear situation, it can be anticipated that the choice of the linear portion of the convective coefficient shall lead to an exact integral transforms solution. On the other hand, for the more general nonlinear situation, other choices of the characteristic coefficient can offer improved convergence with respect to the present choice, by, for instance, including linearized information on the nonlinear portion of the original convection coefficient. The present choice is equivalent to linearizing this term around the steady-state solution ($T = 0$).

The resulting problem with transformed coefficients, after incorporation of the characteristic convective term into the generalized diffusion term, is obtained with the following correspondence:

$$u(x) = u_0 \tag{22a}$$

$$\hat{k}(x) = e^{-u_0 x} \tag{22b}$$

$$\hat{w}(x) = \hat{k}(x) \tag{22c}$$

$$\hat{d}(x) = 0 \tag{22d}$$

$$\hat{P}(x,t,T) = P(x,t,T)\,\hat{k}(x) \tag{22e}$$

and the corresponding eigenvalue problem is given by

$$\frac{d}{dx}\left[\hat{k}(x)\frac{d\psi(x)}{dx}\right] + \mu^2\hat{w}(x)\psi(x) = 0, \quad 0 < x < 1 \tag{23a}$$

with boundary conditions

$$\psi(0) = 0 \tag{23b}$$

$$\psi(1) = 0 \tag{23c}$$

The integral transform pair is then constructed as

$$T(x,t) = \sum_{i=1}^{\infty}\frac{1}{N_i}\psi_i(x)\bar{T}_i(t) \quad \text{inverse} \tag{24a}$$

$$\bar{T}_i(t) = \int_0^1 \hat{w}(x)\psi_i(x)T(x,t)dx \quad \text{transform} \tag{24b}$$

with the normalization integral given by

$$N_i = \int_0^1 \hat{w}(x)\psi_i^2(x)dx \tag{24c}$$

The integral transformation procedure then leads to the following transformed system of ordinary differential equations:

$$\frac{d\bar{T}_i(t)}{dt} + \mu_i^2\bar{T}_i(t) = \bar{g}_i\left(t,\bar{T}_j\right), \quad t > 0 \tag{25a}$$

$$\bar{T}_i(0) = \bar{f}_i \equiv \int_0^1 \hat{w}(x)\psi_i(x)dx, \quad i,j = 1,2,\ldots \tag{25b}$$

while the transformed source term is evaluated from

$$\bar{g}_i\left(t,\bar{T}_j\right) = \int_0^1 P(x,t,T)\psi_i(x)dx \equiv -b\int_0^1 T\frac{\partial T}{\partial x}\psi_i(x)dx \tag{26a}$$

and after substituting the inversion formula into the transformed source term

$$\bar{g}_i(t,\bar{\mathbf{T}}) = -b\sum_{j=1}^{\infty}A_{ij}(\bar{\mathbf{T}})\bar{T}_j(t) \tag{26b}$$

$$A_{ij}(\bar{\mathbf{T}}) = \frac{1}{N_j}\sum_{k=1}^{\infty}A_{ijk}^*\bar{T}_k(t) \tag{26c}$$

$$A_{ijk}^* = \int_0^1 \psi_k(x)\frac{\partial\psi_j(x)}{\partial x}\psi_i(x)dx \tag{26d}$$

The eigenvalue problem with space variable coefficients in Eqs. (23) can be readily solved by the GITT itself, considering a simpler auxiliary eigenvalue problem, and upon integral transformation, yielding an algebraic eigenvalue problem [8,15–17]. The transformed ODE system, for the more general nonlinear situation, can be numerically solved by making use of automatic routines for stiff initial value problems, with automatic accuracy control, such as in the built-in function *NDSolve* of the *Mathematica* package [28]. The hybrid numerical–analytical solution is then readily recovered through the inverse formula, Eq. (24*a*). In the particular case of a linear convection–diffusion problem, $b = 0$ in Eq. (20*e*), the nonlinear source term vanishes, and an exact solution of the transformed ODE system, Eqs. (25), is obtained, and the final solution is given explicitly as

$$T(x,t) = \sum_{i=1}^{\infty}\frac{1}{N_i}\psi_i(x)\bar{f}_i\ e^{-\mu_i^2 t} \tag{27}$$

If a purely diffusive eigenvalue problem was considered instead, such as in Refs. [8] and [18], it would be equivalent to choosing the characteristic convective coefficient as $u(x) = 0$, and all of the rest of the solution procedure would remain the same. In this case, the eigenvalue problem would become

$$\begin{gathered}\frac{d^2\psi(x)}{dx^2} + \mu^2\psi(x) = 0, \quad 0 < x < 1 \\ \psi(0) = 0, \quad \psi(1) = 0\end{gathered} \tag{28a,b}$$

which has a straightforward analytical solution, while the transformed system would still have the form of Eqs. (25), but with the following transformed source term and initial condition:

$$\bar{g}_i(t,\bar{\mathbf{T}}) = -\sum_{j=1}^{\infty}(u_0 B_{ij} + bA_{ij}(\bar{\mathbf{T}}))\bar{T}_j(t) \tag{29a}$$

$$B_{ij} = \frac{1}{N_j}\int_0^1 \frac{\partial\psi_j(x)}{\partial x}\psi_i(x)dx \tag{29b}$$

$$A_{ij}(\bar{\mathbf{T}}) = \frac{1}{N_j}\sum_{k=1}^{\infty}A_{ijk}^*\bar{T}_k(t) \tag{29c}$$

$$A_{ijk}^* = \int_0^1 \psi_k(x)\frac{\partial\psi_j(x)}{\partial x}\psi_i(x)dx \tag{29d}$$

$$\bar{f}_i \equiv \int_0^1 \psi_i(x)dx \tag{29e}$$

with the eigenfunctions, eigenvalues, and norms being obtained from problem (28). It can be observed that in the case when the linearized characteristic convective term is not accounted for in the eigenvalue problem, the transformed system remains coupled even for the linear convection–diffusion problem.

Two-Dimensional Burgers Equation. In order to illustrate the proposed approach in a multidimensional situation, we borrow an example of the two-dimensional Burgers equation, again allowing for direct comparison against previously published results through the GITT with a purely diffusive eigenfunction basis [31]. The problem here considered is written as

$$\frac{\partial T(x,y,t)}{\partial t} + U_x(T)\frac{\partial T(x,y,t)}{\partial x} + U_y(T)\frac{\partial T(x,y,t)}{\partial y} = \frac{\partial^2 T(x,y,t)}{\partial x^2} + \frac{\partial^2 T(x,y,t)}{\partial y^2},\ 0<x<1,\ 0<y<1,\ t>0 \tag{30a}$$

with initial and boundary conditions given by

$$T(x,y,0) = 1,\ 0 \le x \le 1,\ 0 \le y \le 1 \tag{30b}$$

$$T(0,y,t) = 0,\ t>0 \tag{30c}$$

$$T(1,y,t) = 0,\ t>0 \tag{30d}$$

$$T(x,0,t) = 0,\ t>0 \tag{30e}$$

$$T(x,1,t) = 0,\ t>0 \tag{30f}$$

and, for the present application, the nonlinear functions $U_x(T)$ and $U_y(T)$ are taken as

$$U_x(T) = u_0 + b_x T \tag{30g}$$

$$U_y(T) = v_0 + b_y T \tag{30h}$$

Again, from direct comparison against the multidimensional formulations in Eqs. (16)–(18), taking the linear portion of the velocity coefficients to represent the characteristic convective terms and transporting the remaining terms to the nonlinear source term, the following coefficients' correspondence can be reached:

$$u_x(\mathbf{x}) = u_0 \tag{31a}$$

$$v_x(\mathbf{x}) = v_0 \tag{31b}$$

$$w(\mathbf{x}) = 1 \tag{31c}$$

$$k(\mathbf{x}) = 1 \tag{31d}$$

$$d(\mathbf{x}) = 0 \tag{31e}$$

$$P(\mathbf{x},t,T) = -\left(b_x T\frac{\partial T}{\partial x} + b_y T\frac{\partial T}{\partial y}\right) \tag{31f}$$

$$\hat{k}_x(\mathbf{x}) = e^{-u_0 x} \tag{31g}$$

$$\hat{k}_y(\mathbf{x}) = e^{-v_0 y} \tag{31h}$$

$$\hat{k}(\mathbf{x}) = \hat{k}_x(x)\hat{k}_y(y) = e^{-(u_0 x + v_0 y)} \tag{31i}$$

$$\hat{w}(\mathbf{x}) = \hat{k}(\mathbf{x}) \tag{31j}$$

$$\hat{d}(\mathbf{x}) = 0 \tag{31k}$$

$$\hat{P}(\mathbf{x},t,T) = P(\mathbf{x},t,T)\,\hat{k}(\mathbf{x}) \tag{31l}$$

The resulting transformed equation with the generalized diffusion terms representation is then given by

$$\hat{w}(x,y)\frac{\partial T(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\left[\hat{k}(x,y)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[\hat{k}(x,y)\frac{\partial T}{\partial y}\right] + \hat{P}(\mathbf{x},t,T),\ 0<x<1,\ 0<y<1,\ t>0 \tag{32}$$

which leads to the following eigenvalue problem:

$$\frac{\partial}{\partial x}\left[\hat{k}(x,y)\frac{\partial \psi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\hat{k}(x,y)\frac{\partial \psi}{\partial y}\right] + \mu^2 \hat{w}(x,y)\psi(x,y) = 0,\ 0<x<1,\ 0<y<1 \tag{33a}$$

with boundary conditions

$$\psi(0,y) = 0 \tag{33b}$$

$$\psi(1,y) = 0 \tag{33c}$$

$$\psi(x,0) = 0 \tag{33d}$$

$$\psi(x,1) = 0 \tag{33e}$$

This two-dimensional eigenvalue problem with space variable coefficients is also readily solved by applying the GITT itself, based on a simpler auxiliary eigenvalue problem, as described in different sources [8,15–17]. The associated integral transform pair is written as

$$T(x,y,t) = \sum_{i=1}^{\infty} \frac{1}{N_i}\psi_i(x,y)\bar{T}_i(t) \quad \text{inverse} \tag{34a}$$

$$\bar{T}_i(t) = \int_0^1\int_0^1 \hat{w}(x,y)\psi_i(x,y)T(x,y,t)dydx \quad \text{transform} \tag{34b}$$

with the normalization integral given by

$$N_i = \int_0^1\int_0^1 \hat{w}(x,y)\psi_i^2(x,y)dydx \tag{34c}$$

The following transformed system of ordinary differential equations is then obtained:

$$\frac{d\bar{T}_i(t)}{dt} + \mu_i^2 \bar{T}_i(t) = \bar{g}_i\left(t,\bar{T}_j\right),\ \ t>0 \tag{35a}$$

$$\bar{T}_i(0) = \bar{f}_i \equiv \int_0^1\int_0^1 \hat{w}(x,y)\psi_i(x,y)dydx,\ \ i,j = 1,2,\ldots \tag{35b}$$

while the transformed source term is evaluated from

$$\bar{g}_i\left(t,\bar{T}_j\right) = \int_0^1\int_0^1 \hat{P}(x,y,t,T)\psi_i(x,y)dydx \equiv -\int_0^1\int_0^1 \hat{k}(x,y)T(x,y,t)\left(b_x\frac{\partial T}{\partial x} + b_y\frac{\partial T}{\partial y}\right)\psi_i(x,y)dydx \tag{36a,b}$$

and after substituting the inversion formula into the transformed source term

$$\bar{g}_i(t,\bar{\mathbf{T}}) = -\sum_{j=1}^{\infty} A_{ij}(\bar{\mathbf{T}})\bar{T}_j(t) \quad (36c)$$

$$A_{ij}(\bar{\mathbf{T}}) = \frac{1}{N_j}\sum_{k=1}^{\infty} A^*_{ijk}\bar{T}_k(t) \quad (36d)$$

$$A^*_{ijk} = \int_0^1\int_0^1 \hat{k}(x,y)\psi_k(x,y)\left(b_x\frac{\partial\psi_j}{\partial x} + b_y\frac{\partial\psi_j}{\partial y}\right)\psi_i(x,y)dydx \quad (36e)$$

Also here, the linear convection–diffusion problem is readily solved in exact form, since the nonlinear source term vanishes, yielding a decoupled transformed ODE system, and the final solution in this case is recovered as

$$T(x,y,t) = \sum_{i=1}^{\infty}\frac{1}{N_i}\psi_i(x,y)\bar{f}_i\ e^{-\mu_i^2 t} \quad (37)$$

For the more general nonlinear situation, system (35) can be again accurately solved with a numerical routine for stiff ODE systems, with automatic accuracy and precision controls, such as the built-in function NDSolve of the Mathematica package [28], and the numerical solution for the transformed potentials is provided as interpolated continuous functions along the t variable. Then, the inverse formula (34a) is recalled to provide the desired hybrid numerical–analytical solution for the original potential anywhere within the domain.

Here also, the traditional solution via a purely diffusive eigenvalue problem [31] can be easily recovered, by adopting $u_x(\mathbf{x}) = 0$ and $v_x(\mathbf{x}) = 0$, and carrying the full convective terms to the source term, i.e., $P(x,y,t) = -[U_x(T)(\partial T(x,y,t)/\partial x) + U_y(T)(\partial T(x,y,t)/\partial y)]$. Although the transformed system (35) remains the same, the transformed source terms and initial conditions need to be modified accordingly. Similarly to the one-dimensional situation, when choosing the purely diffusive eigenvalue problem, the transformed ODE system remains coupled even for the linear convection–diffusion situation, and an explicit solution such as in Eq. (37) is not obtainable in this case.

Results and Discussion

The inspection of the convergence rates improvement achieved through the proposed convective eigenfunction expansions is now illustrated. A few cases with representative numerical values of the governing parameters have been considered, to offer a set of numerical results in both one- and two-dimensional test cases.

First, the one-dimensional convection–diffusion problem is analyzed. Tables 1 and 2 illustrate the convergence behavior of the eigenfunction expansions for $T(x,t)$, respectively, with $u_0 = 10$, $b = 0$ and $u_0 = 20$, $b = 0$ (linear problem), while Tables 3 and 4 provide results for the cases, respectively, with $u_0 = 10$, $b = 5$ and $u_0 = 1$, $b = 10$ (nonlinear problem). In Tables 1 and 2, for the linear situation, a maximum truncation order of $I = 100$ terms has been considered in the solution of the generalized diffusive eigenvalue problem and $I < 100$ in computing the $T(x,t)$ expansion. From the columns associated with the use of the convective eigenvalue problem (labeled "Conv."), one may clearly observe the marked gain in convergence rates in comparison to the diffusive eigenfunction basis (labeled "Diff."), when the convective eigenvalue problem fully accounts for the influence of the linear convection term. From the first two columns in Tables 1 and 2, for the larger value of $t = 0.05$, it can be observed that the results obtained through the convective eigenvalue problem are already fully converged to four significant digits at truncation orders as low as $I = 4$, while the solution that employs the diffusive eigenvalue problem needs more than $I = 40$ terms to achieve the same level of precision. For the smaller value of $t = 0.01$, slightly larger truncation orders are required, a typical behavior in eigenfunction expansions. In both cases (Tables 1 and 2), for the convective basis, full convergence to four significant digits has already been achieved for $I = 10$, while truncation orders of around $I = 70$ are required through the purely diffusive basis. The two solutions perfectly match each other, but in this linear case, the computational cost of the convective basis solution, which is fully explicit and analytical, is indeed negligible in comparison to the coupled transformed system solution via the traditional diffusive eigenvalue problem choice.

For the nonlinear one-dimensional problem, a maximum truncation order of $I = 30$ terms has been considered in the solution of the generalized diffusive eigenvalue problem and $I < 30$ in computing the potential expansion. From Tables 3 and 4 for the nonlinear situation, with $u_0 = 10$, $b = 5$ and $u_0 = 1$, $b = 10$, respectively, one concludes that similar observations can be drawn with respect to the comparative behavior of the convective and diffusive basis, with a somehow less marked difference in this case,

Table 1 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 1D Burgers equation in linear formulation

	$u_0 = 10$, $b = 0$ (linear problem)					
I	T(0.5,0.05) Conv.	T(0.5,0.05) Diff.	T(0.1,0.01) Conv.	T(0.1,0.01) Diff.	T(0.9,0.01) Conv.	T(0.9,0.01) Diff.
2	0.3865	0.4624	0.1637	0.2659	−1.814	0.4302
4	0.3797	0.3562	0.2693	0.2892	0.1994	0.6538
6	0.3797	0.3888	0.2854	0.2803	0.7033	0.7263
8		0.3754	0.2862	0.2818	0.7369	0.7484
10		0.3820	0.2862	0.2843	0.7374	0.7508
12		0.3783		0.2859	0.7374	0.7464
14		0.3806		0.2867		0.7411
16		0.3791		0.2870		0.7372
18		0.3801		0.2868		0.7353
20		0.3794		0.2866		0.7350
30		0.3798		0.2861		0.7382
40		0.3797		0.2863		0.7371
50		0.3797		0.2862		0.7376
60				0.2862		0.7373
70						0.7375
80						0.7374
90						0.7374

Table 2 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 1D Burgers equation in linear formulation

	$u_0 = 20$, $b = 0$ (linear problem)					
I	$T(0.9,0.05)$ Conv.	$T(0.9,0.05)$ Diff.	$T(0.1,0.01)$ Conv.	$T(0.1,0.01)$ Diff.	$T(0.5,0.01)$ Conv.	$T(0.5,0.01)$ Diff.
2	0.1874	0.0805	0.0517	0.1573	2.8290	1.0472
4	0.2375	0.1786	0.1040	0.0901	0.5869	0.9374
6	0.2375	0.2289	0.1140	0.0937	1.0064	1.0013
8		0.2475	0.1145	0.1075	0.9737	0.9601
10		0.2500	0.1145	0.1148	0.9749	0.9837
12		0.2462		0.1176	0.9749	0.9692
14		0.2412		0.1178		0.9787
16		0.2374		0.1168		0.9722
18		0.2355		0.1155		0.9768
20		0.2351		0.1145		0.9734
30		0.2384		0.1145		0.9753
40		0.2372		0.1145		0.9747
50		0.2377				0.9750
60		0.2374				0.9748
70		0.2376				0.9749
80		0.2375				0.9749
90		0.2376				

Table 3 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 1D Burgers equation in nonlinear formulation

	$u_0 = 10$, $b = 5$ (nonlinear problem)					
I	$T(0.5,0.05)$ Conv.	$T(0.5,0.05)$ Diff.	$T(0.1,0.01)$ Conv.	$T(0.1,0.01)$ Diff.	$T(0.9,0.01)$ Conv.	$T(0.9,0.01)$ Diff.
2	0.2862	0.3606	0.1492	0.2340	−1.437	0.4504
4	0.2762	0.2496	0.2331	0.2343	0.4146	0.6881
6	0.2770	0.2877	0.2439	0.2336	0.7687	0.7808
8	0.2768	0.2718	0.2447	0.2398	0.7980	0.8125
10	0.2769	0.2795	0.2448	0.2435	0.8009	0.8162
12	0.2769	0.2753	0.2448	0.2452	0.8006	0.8104
14		0.2778	0.2448	0.2458	0.8001	0.8035
16		0.2762	0.2447	0.2458	0.7998	0.7984
18		0.2773	0.2447	0.2455	0.7996	0.7960
20		0.2765		0.2451	0.7996	0.7956
22		0.2771		0.2448		0.7965
24		0.2766		0.2446		0.7977
26		0.2770		0.2446		0.7988
28		0.2767		0.2446		0.7995
30		0.2769		0.2447		0.7996

Table 4 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 1D Burgers equation in nonlinear formulation

	$u_0 = 1$, $b = 10$ (nonlinear problem)					
I	$T(0.5,0.05)$ Conv.	$T(0.5,0.05)$ Diff.	$T(0.1,0.01)$ Conv.	$T(0.1,0.01)$ Diff.	$T(0.9,0.01)$ Conv.	$T(0.9,0.01)$ Diff.
2	0.6245	0.6409	0.2931	0.2897	0.4189	0.4141
4	0.5656	0.5616	0.3653	0.3599	0.6283	0.6204
6	0.5752	0.5763	0.3767	0.3742	0.6804	0.6789
8	0.5735	0.5731	0.3798	0.3789	0.6904	0.6914
10	0.5738	0.5740	0.3803	0.3799	0.6912	0.6924
12	0.5738	0.5737	0.3803	0.3801	0.6908	0.6916
14		0.5738	0.3803	0.3802	0.6906	0.6909
16		0.5737	0.3802	0.3803	0.6905	0.6905
18		0.5738	0.3802	0.3803	0.6904	0.6903
20		0.5738		0.3803	0.6904	0.6903
22				0.3803		0.6903
24				0.3802		0.6904
26				0.3802		0.6905
28				0.3802		0.6905
30						0.6905

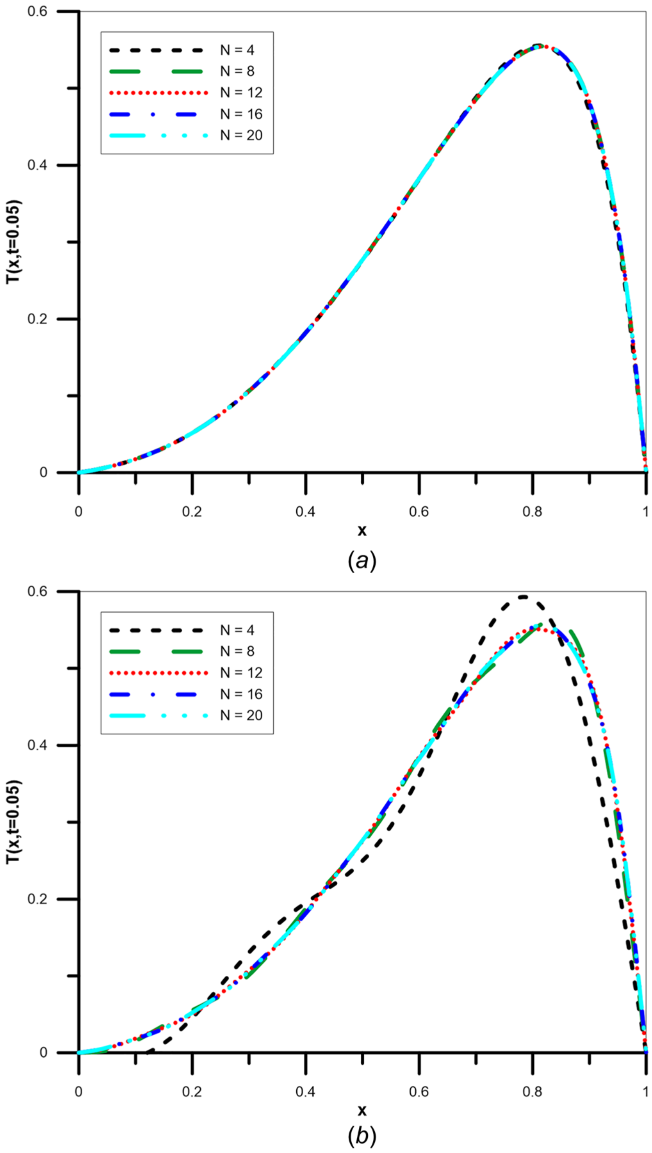

Fig. 1 (*a*) and (*b*) Convergence behavior of *T(x,t)* for nonlinear 1D Burgers equation at $t = 0.05$ for truncation orders $I = 4$, 8, 12, 16, and 20: (*a*) convective basis and (*b*) diffusive basis

since for the nonlinear case, the convective eigenvalue problem does not fully account for the convective term influence, but only for a characteristic behavior of the velocity coefficient, here taken as the linear part of the velocity expression (u_0). For instance, in Table 3, at $x = 0.5$ and $t = 0.05$, convergence to four significant digits is achieved for I as low as 10 for the convective basis, and the diffusive basis requires $I = 30$, while for $x = 0.9$ and $t = 0.01$, the convective basis yields four significant digits at $I = 18$, and the diffusive basis requires at least $I = 30$ terms. In Table 4, as the linear component of velocity is decreased (u_0, from 10 to 1), and the nonlinear coefficient is increased (b, from to 5 to 10), it can be observed a slight improvement in the diffusive basis convergence, while the convective basis retains basically the same convergence behavior.

Figures 1(*a*) and 1(*b*) provide a graphical comparison between the convergence rates achievable by the eigenfunction expansions with the convective and diffusive basis, respectively, for the nonlinear one-dimensional Burgers equation, with $u_0 = 10$ and $b = 5$, at $t = 0.05$ and truncation orders $I = 4$, 8, 12, 16, and 20.

Clearly, the convective basis solutions practically coincide to the graph scale, for all the considered truncation orders, even the lowest order $I = 4$, while the solutions obtained via the diffusive basis are still not graphically converged for $I = 4$, 8, and 12, but practically merged for $I = 16$ and 20.

Tables 5–7 provide a convergence analysis for $T(x,y,t)$ in the two-dimensional Burgers equation, for $b_x = 0$, $b_y = 0$ (linear problem), respectively, with, $u_0 = 1$ and $v_0 = 1$, $u_0 = 10$ and $v_0 = 1$, $u_0 = 10$ and $v_0 = 10$. For this linear two-dimensional problem, a maximum truncation order of $I = 160$ terms has been considered in the solution of the generalized diffusive eigenvalue problem and $I < 160$ in computing the potential expansion. The solution of the eigenvalue problems, either for the convective or diffusive basis, employs a reordering scheme based on the sum of the squared eigenvalues of the auxiliary problem [11,31]. The results in Tables 5–7 again reconfirm the excellent convergence behavior of the expansions that follow the convective basis proposal, with four converged significant digits, for instance in Table 6, at truncation orders as low as $I = 50$ ($x = 0.5$, $y = 0.1$, $t = 0.01$), $I = 50$ ($x = 0.1$, $y = 0.1$, $t = 0.01$), and $I = 20$ ($x = 0.9$, $y = 0.1$, $t = 0.05$), as expected, with decreasing required truncation orders for larger values of the variable t. In this case, the results achieved by the diffusive basis are only fully converged for the largest truncation order here adopted, $I = 160$, but it is clearly noticeable that this solution presents an oscillatory behavior to reach convergence at the fourth significant digit. Again, it should be recalled that a fully analytical solution is achieved through the use of the convective basis in this linear example, as given by Eq. (37).

Table 5 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 2D Burgers equation in linear formulation

	$u_0 = 1$, $v_0 = 1$, $b_x = 0$, $b_y = 0$ (linear problem)					
I	T(0.5,0.1,0.01) Conv.	T(0.5,0.1,0.01) Diff.	T(0.1,0.1,0.01) Conv.	T(0.1,0.1,0.01) Diff.	T(0.9,0.1,0.05) Conv.	T(0.9,0.1,0.05) Diff.
10	0.4893	0.4910	0.2079	0.2118	0.05874	0.05889
20	0.4944	0.5009	0.2391	0.2440	0.05876	0.05881
30	0.4952	0.4955	0.2450	0.2451	0.05876	0.05873
40	0.4959	0.4964	0.2462	0.2469		0.05878
50	0.4959	0.4953	0.2463	0.2457		0.05876
60	0.4960	0.4954	0.2464	0.2458		0.05877
70	0.4960	0.4954	0.2464	0.2458		0.05876
80		0.4954		0.2458		0.05876
90		0.4953		0.2458		0.05876
100		0.4953		0.2459		0.05873
110		0.4956		0.2460		0.05876
120		0.4956		0.2460		0.05876
130		0.4956		0.2460		0.05875
140		0.4956		0.2461		0.05873
150		0.4959		0.2463		0.05876
160		0.4959		0.2463		0.05876

Table 6 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 2D Burgers equation in linear formulation

	$u_0=10$, $v_0=1$, $b_x=0$, $b_y=0$ (linear problem)					
I	T(0.5,0.1,0.01) Conv.	T(0.5,0.1,0.01) Diff.	T(0.1,0.1,0.01) Conv.	T(0.1,0.1,0.01) Diff.	T(0.9,0.1,0.05) Conv.	T(0.9,0.1,0.05) Diff.
10	0.5188	0.4856	0.1114	0.1334	0.1082	0.09789
20	0.4942	0.5014	0.1357	0.1550	0.1083	0.1023
30	0.4958	0.4978	0.1411	0.1379	0.1083	0.1062
40	0.4954	0.4924	0.1420	0.1450		0.1085
50	0.4954	0.4914	0.1422	0.1393		0.1094
60	0.4955	0.4941	0.1422	0.1406		0.1098
70	0.4955	0.4951	0.1423	0.1408		0.1098
80		0.4952	0.1423	0.1408		0.1098
90		0.4936		0.1403		0.1095
100		0.4929		0.1412		0.1091
110		0.4931		0.1417		0.1092
120		0.4942		0.1409		0.1087
130		0.4946		0.1405		0.1088
140		0.4946		0.1417		0.1082
150		0.4949		0.1423		0.1083
160		0.4949		0.1424		0.1084

Table 7 Convergence analysis of eigenfunction expansions with convective and diffusive eigenvalue problems in the solution of the 2D Burgers equation in linear formulation

	$u_0=10$, $v_0=10$, $b_x=0$, $b_y=0$ (linear problem)					
I	T(0.5,0.1,0.01) Conv.	T(0.5,0.1,0.01) Diff.	T(0.1,0.1,0.01) Conv.	T(0.1,0.1,0.01) Diff.	T(0.9,0.1,0.05) Conv.	T(0.9,0.1,0.05) Diff.
10	0.3142	0.2845	0.05472	0.06619	0.01262	0.01670
20	0.2807	0.3257	0.07683	0.10040	0.01264	0.01147
30	0.2855	0.2817	0.08114	0.07001	0.01264	0.01025
40	0.2858	0.2887	0.08188	0.08732		0.01474
50	0.2860	0.2787	0.08208	0.07715		0.01170
60	0.2860	0.2842	0.08211	0.08035		0.01237
70	0.2861	0.2835	0.08212	0.08079		0.01262
80	0.2861	0.2840	0.08212	0.08070		0.01276
90		0.2808		0.07983		0.01238
100		0.2809		0.08090		0.01193
110		0.2837		0.08215		0.01278
120		0.2851		0.08100		0.01272
130		0.2828		0.07964		0.01214
140		0.2825		0.08132		0.01169
150		0.2858		0.08307		0.01275
160		0.2860		0.08245		0.01297

Conclusions

A convergence enhancement technique has been here proposed to be employed in the integral transforms solution of either linear or nonlinear convection–diffusion problems. Based on straightforward exponential coefficients transformation, the original problem formulation is rewritten so as to incorporate into a generalized diffusion term, in part or fully, the contribution of the convective term. Then, this reformulated problem is solved by the GITT, considering an eigenvalue problem that includes the generalized diffusion term, thus somehow accounting for the convective term effects in the eigenfunction expansion basis. This methodological alternative is equivalent to the direct consideration of a convective eigenvalue problem in the solution through GITT, instead of following the traditional path of considering a purely diffusive eigenvalue problem and transporting all of the convective terms to the equation source term. First, a one-dimensional but fairly general nonlinear convection–diffusion formulation is considered, to allow for a basic understanding of the proposed approach. A characteristic velocity coefficient, which for linear problems coincides with the actual original coefficient, is then manipulated to appear within the new coefficients of the reformulated problem, including a new space variable generalized diffusion coefficient. For nonlinear problems, a linear characteristic convective term is selected, and the remaining nonlinear terms are incorporated into the modified nonlinear source term. Next, the multidimensional problem is considered, and again, characteristic coefficients for the convective terms are selected to be incorporated through the exponential transformation into the generalized diffusion terms. Also, it has been discussed the path of avoiding nonself adjoint eigenvalue problems by considering characteristic velocity coefficients that are functions only of the specific space variable in each coordinate direction. Finally, the approach is illustrated for a few test cases involving examples of Burgers equation in both one and two dimensions. It is then demonstrated that the solutions based on the convective eigenvalue problem choice present markedly improved convergence rates in comparison to the traditional solutions with the purely diffusive basis and with the convective terms fully incorporated into the source terms. Clearly, the adoption of a more informative eigenfunction expansion basis, including at least part of the convective term contribution, while at the same time reducing the importance of the source term that incorporates the convective terms, significantly contributes to the convergence acceleration. The approach also opens new perspectives for future work, as research advances toward the integral transform solution of nonlinear problems through the adoption of nonlinear eigenvalue problems [29], when the exponential transformation can be employed to fully incorporate the nonlinear convective terms into a generalized nonlinear diffusion formulation.

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